(3 Hours) **[Total Marks: 100]**

Note: (i) All questions are compulsory.

(ii)Figures to the right indicate marks for respective parts.

Q.1 Choose correct alternative in each of the following (20)

$i.$ Let T: V →V' be a linear transformation. Img(T) is a subspace of

(a) V (b) V'

(c) V and V' (d) None of the above

$ii.$ Let T: V →V' be linear transformation. T^{-1}:V'→ V is also a linear transformation provided

(a) T is injective (b) T is surjective

(c) T is bijective (d) None of the above.

$iii.$ Let T be the linear transformation such that T(1, 0) = (1 , 1) and T(0, 1) = (1 , -1), then T(1,1) =

(a) (2,0) (b) (-14,-4)

(c) (11,5) (d) None of the above.

$iv.$ Det$\begin{pmatrix} a & 0 & 0 & 0 \\ b & b & 0 & 0 \\ c & c & c & 0 \\ d & d & d & d \end{pmatrix}$ is

(a) $a+b+c+d$ (b) $abcd$

(c) $ab^2c^3d^4$ (d) $a+b^2+c^3+d^4$

$v.$ If $A = \begin{pmatrix} 2 & 2 & 3 \\ 5 & 1 & 4 \\ 1 & 2 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 2 & 3 \\ 5 & 1 & 4 \\ 5 & 6 & 7 \end{pmatrix}$ then which of the following is true

(a) $detA = detB$ (b) $detA = 5 + detB$

(c) $detA \neq detB$ (d) none of these

$vi.$ If A be a $n \times n$ matrix with $detA \neq 0$ then

(a) $adjA = detA.I_n$ (b) $adjA = \frac{1}{detA}A^{-1}$

(c) $adjA = (detA)A^{-1}$ (d) none of these

$vii.$ Given that u, v, w are linearly independent vectors of $\mathbb{R}^3$, which of the below is false .

(a) Volume of the parallelepiped obtained by the vectors $u+v\,,v\,,w$ is the same as that obtained by u,v,w

(b) Volume spanned by $3u,v,w$ is the same as spanned by u,v,w

(c) Volume spanned by u,v,w is the determinant of the matrix taking u,v,w as column vectors.

(d) Volume of parallelepiped using vectors $u+v,v,w$ is same as that spanned by $u,v-w,w$.

$viii.$ On $\mathbb{R}^3$, the map $\langle,\rangle : \mathbb{R}^3\times\mathbb{R}^3\to\mathbb{R}$ defined by $\langle x,y\rangle = a_1x_1y_1+a_2x_2y_2+a_3x_3y_3$ where $x=(x_1,x_2,x_3), y=(y_1,y_2,y_3)\in\mathbb{R}^3$ is an inner product if

(a) $a_1,a_2,a_3,\geq 0$

(b) a_1,a_2,a_3 are positive real numbers

(c) a_1,a_2,a_3 are non-zero

(d) At least one of a_1,a_2,a_3 is non-zero

$ix.$ Consider the following sets of vectors in $\mathbb{R}^2$ under dot product

(i) $S_1=\{(0,1),(2,0)\}$

(ii) $S_2=\left\{\left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right),\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)\right\}$

(iii) $S_3=\{(0,-1),(0,1)\}$

(iv) $S_4=\left\{\left(\frac{1}{\sqrt{2}},\frac{1}{2}\right),\left(-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)\right\}$

(a) All the sets are orthogonal sets

(b) S_2 and S_4 are orthogonal sets

(c) S_1 and S_2 are orthogonal sets

(d) S_1 and S_3 are orthogonal sets

$x.$ Let $v=(a,b)\neq 0$ in $\mathbb{R}^2$. The set of all vectors orthogonal to v in $\mathbb{R}^2$ represents

(a) A straight line passing through origin and v

(b) A straight line passing through origin and perpendicular to v

(c) Empty set

(d) None of these

Q2. Attempt any **ONE** question from the following: (08)

$a)$ i. State and prove Rank-Nullity theorem.

ii. Prove that the solution of non-homogenous system AX=B has a solution if and only if rank A= rank[A|B].

Q.2 Attempt any **TWO** questions from the following: (12)

b) i. Write all three types of elementary matrices. Give an example in each case.

ii. If $T:V\to W$ is linear transformation, $B=\{v_1, v_2, \ldots\ldots v_n\}$ is linearly independent subset of V and $\ker T=\{0\}$ then show that $\{T(v_1), T(v_2), \ldots\ldots T(v_n)\}$ is linearly independent subset of W.

iii. Find the rank of the matrix A where $A = \begin{bmatrix} 1 & 1 & 0 & -1 \\ -2 & 1 & 0 & 0 \\ 3 & 2 & 0 & 1 \\ -1 & 0 & 1 & 1 \end{bmatrix}$.

iv. Let $T: \mathbb{R}^3 \to \mathbb{R}^2$, $T(x,y,z)=(x+z,2x+y)$ be a linear transformation. Find the matrix of this linear transformation with respect to standard basis.

Q3. Attempt any **ONE** question from the following: (08)

a) i. Let $A^1, A^2 \in \mathbb{R}^2$ and $k \in \mathbb{R}$. Show that
I) $\det(A^1, A^2) = 0$ iff $\{A^1, A^2\}$ is linearly dependent.
II) $\det(A^1, A^2 + kA^1) = \det(A^1, A^2)$.

ii. State and prove the Cramer's rule $for\ n \times n\ linear\ system\ AX = b$.

Q3. Attempt any **TWO** questions from the following: (12)

b) i. Write the definition of determinant for 2×2 $matrices$. Find the determinant of the following matrix using definition of determinant
$$A = \begin{pmatrix} 2 & 3 \\ 5 & 2 \end{pmatrix}$$

ii. Prove that Area of a parallelogram spanned by $(x_1, x_2)\&(y_1, y_2)$ is $\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}$

iii. State Cramer's Rule for a $n \times n$ linear system. Use it to find solution to
$$\begin{pmatrix} 1 & 4 & 2 \\ 3 & -3 & 6 \\ 2 & 0 & 5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ -4 \end{pmatrix}$$

iv. State the result for inverse of a matrix in terms of its adjoint.

Find A^{-1} for $A = \begin{pmatrix} 1 & 1 & 3 \\ 0 & -1 & 2 \\ 1 & 2 & -1 \end{pmatrix}$ using adjoint.

Q4. Attempt any **ONE** question from the following: (08)

a) i. Define norm of a vector in inner product space V. Let V be an inner product space. For $x, y \in V$, show that

(p) $\|\alpha x\| = |\alpha|\|x\|$

(q) $\|x - y\|^2 = \|x\|^2 + \|y\|^2 - 2\|x\|\|y\| \cos\theta$
where θ is the angle between x & $y, x \neq 0, y \neq 0$.

(r) $|\|x\| - \|y\|| \leq \|x - y\|$

ii. State and prove Gram-Schmidt Orthogonalisation Process.

Q4. Attempt any **TWO** questions from the following: (12)

b) i. Define orthogonal vectors in a real inner product space V. If$\{v_i\}_{i=1}^{n}$ is a set of pair wise orthogonal vectors in V, then show $\|\sum_{i=1}^{n} v_i\|^2 = \sum_{i=1}^{n}\|v_i\|^2$.

ii. Prove that the function given by $\langle x, y\rangle = x_1y_1 + 2x_1y_2 + 2x_2y_1 + 5x_2y_2$ is an inner product on $\mathbb{R}^2$, where $x = (x_1, x_2),\ y = (y_1, y_2) \in \mathbb{R}^2$.

iii. Define angle between two vectors in a real inner product space. Find angle between $p(x) = 2x^2 + 1$ and $q(x) = x^2 + 2x + 1$with respect to the inner product $\langle p, q\rangle = a_0b_0 + a_1b_1 + a_2b_2$
where $p(x) = a_0 + a_1x + a_2x^2$ and $q(x) = b_0 + b_1x + b_2x^2$.

iv. Define $W^{\perp}$, the orthogonal compliment of a subspace W of a real inner product space V. Show that $W^{\perp}$ is a subspace of V.

Q5. Attempt any **FOUR** questions from the following: (20)

a) Let AX=0 be a homogeneous system with m equations and n unknowns. Prove that the set of solutions of this homogeneous system AX=0 forms a subspace of $\mathbb{R}^n$.

b) Verify Rank-Nullity theorem for the following linear transformation T:$\mathbb{R}^3 \rightarrow \mathbb{R}^3$
T(x,y,z)=(x+y,y+z,x+z).

c) Let $A \in M_n(\mathbb{R})$ then prove that
$AX = b$ has a unique solution for each $b \in \mathbb{R}^n \iff detA \neq 0$

$d)$ State the Laplace expansion formula for determinant.
Use Laplace expansion to find the determinant of the following matrix

$$\begin{pmatrix} 2 & 2 & 3 & 3 \\ 3 & 2 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 2 & 0 & 1 & 1 \end{pmatrix}$$

$e)$ Find the projection of $v = (1,2,3)$ on $u = (-2,1,0)$ with respect to the inner product where $\langle u, v\rangle = 2x_1y_1 + x_2y_2 + 3x_3y_3$, $u = (x_1, x_2, x_3)$& $v = (y_1, y_2, y_3)$. Also verify Cauchy-Schwarz inequality for u & v.

$f)$ Prove that a parallelogram is a rhombus if and only if the diagonals are perpendicular to each other.

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