(3 Hours) **[Total Marks: 100]**

Note: (i) All questions are compulsory.
(ii)Figures to the right indicate marks for respective parts.

Q.1 Choose correct alternative in each of the following (20)

$i.$ Rank Nullity Theorem states that if $T:V\to W$ is a linear transformation, then

(a) Dim V = dim(ImT) + dim$W$ (b) Dim V - dim(ImT) = dim (ker T)
(c) Dim V = dim W (d) Dim V/W = dim$V$ - dim W

$ii.$ Let T be the linear transformation defined by T(**x**)=**Ax**, where **A**= $\begin{pmatrix}5 & 1\\ 2 & 3\end{pmatrix}$.

What is the image of $\begin{pmatrix}2\\ 1\end{pmatrix}$?

(a) $\begin{pmatrix}11\\ 7\end{pmatrix}$ (b) $\begin{pmatrix}11\\ 13\end{pmatrix}$
(c) $\begin{pmatrix}7\\ 5\end{pmatrix}$ (d) None of these

$iii.$ Which of the following is a linear transformation from $T:\mathbb{R}^2\to\mathbb{R}^2$,?

(a) $T(x,y)=(x-y\ ,y)$ (b) $T(x,y)=(|x|\ , y+1)$
(c) $T(x,y)=(x^2+y, x-y)$ (d) All the above

$iv.$ Let A be a $m\times n$ matrix and let row rank = p and column rank = q. Then

(a) p = q (b) p > q
(c) p < q (d) None of the above

$v.$ If $A=\begin{pmatrix}1 & 2 & 1\\ 3 & 2 & 4\\ 5 & 6 & 1\end{pmatrix}$ and $B=\begin{pmatrix}4 & 8 & 4\\ 3 & 2 & 4\\ 5 & 6 & 1\end{pmatrix}$ then which of the following is true?

(a) $detA = detB$ (b) $detA = 4detB$
(c) $4detA = detB$ (d) None of these

$vi.$ $Det(2e_{1,}, e_1+3e_2, -e_3)$ where e_1,e_2, e_3 are standard basis elements of $\mathbb{R}^3$ is

(a) 3 (b) 1
(c) 6 (d) -6

$vii.$ Cramer's rule is used to

(a) Find solution of homogeneous system of linear equations (b) Find solution of non- homogeneous system of linear equations
(c) Find determinant of the matrix (d) Find inverse of the matrix

$viii.$ Let V be a real inner product space and $x,y\in V$. If $\|x\|=\|y\|$, then

(a) $x+y$ and $x-y$ orthogonal (b) $x=y$
(c) x and y are orthogonal (d) None of these

ix. Let $\{v_1, v_2, \ldots, v_n\}$ be an orthonormal basis of an inner product space $x = \sum_i x_i v_i$. Then $\|x\|^2 =$

(a) $\sum_{i=1}^{n} \langle x, x_i \rangle^2$ (b) $\sum_{i=1}^{n} \langle v_i, x_i \rangle^2$

(c) $\sum_{i=1}^{n} \langle x, v_i \rangle^2$ (d) None of these

x. Let $v = (a, b) \neq (0,0)$ in $\mathbb{R}^2$. The set of all vectors orthogonal to v in $\mathbb{R}^2$ represents

(a) A straight line passing through origin and v (b) A straight line passing through origin and perpendicular to v

(c) Empty set (d) None of these

Q2. Attempt any **ONE** question from the following: (08)

a) i. Define p) Linear Transformation q) kernel of a linear transformation. Further prove that if $T : V \to V'$ is a linear transformation then T is injective if and only if ker $T = \{0\}$.

ii. Prove: Let $A \in M_n(\mathbb{R})$ then prove that the system of homogenous system of n linear equations in n unknowns, $AX = 0$ has only the trivial solution if and only if Rank(A) = n.

Q.2 Attempt any **TWO** questions from the following: (12)

b) i. If $F : \mathbb{R}^3 \to \mathbb{R}^3$ such that $F(x,y,z) = (x+y+z, x+2y-z, 3x+5y-z)$, show that F is linear transformation. Find whether F is non-singular.

ii. If $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that $T(x,y,z) = (x, 2y, 0)$, find ker T, basis of ker T and nullity T.

iii. Show that A and B are row equivalent matrices where

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 1 & 1 \end{pmatrix}, B = \begin{pmatrix} 3 & 0 & 1 \\ 0 & 3 & 1 \end{pmatrix}$$

iv. Test for consistency and if possible solve the following system.

$2x - y + z = 9, 3x - y + z = 6, 4x - y + 2z = 7, -x + y - z = 4$

Q3. Attempt any **ONE** question from the following: (08)

a) i. Let $A \in M_n(\mathbb{R})$. Prove that $AX = 0$ has a non trivial solution if and only if $det\ A = 0$.

Further check whether the homogeneous system

$$\begin{pmatrix} 1 & 2 & 0 \\ -1 & 0 & 1 \\ 5 & 3 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$ has a non trivial solution.

ii. Let $v_1, v_2, \ldots, v_n \in \mathbb{R}^n$. Show that
I) If $\{v_1, v_2, \ldots, v_n\}$ is linearly dependent then $\det(v_1, v_2, \ldots, v_n) = 0$.
II) $\det(v_1, .., v_i, .., v_j, .., v_n) = -\det(v_1, .., v_j, .., v_i, .., v_n)$
for $1 \leq i \neq j \leq n$

Q3. b) Attempt any **TWO** questions from the following: (12)

i. Define adjoint of a matrix. Find $\boldsymbol{A}^{-1}$ for $\boldsymbol{A} = \begin{pmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{pmatrix}$ using adjoint.

ii. Define bilinear map.
Further check whether the following map is bilinear.
$f: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ such that $f\big((a,b),(c,d)\big) = ab + cd + 1$.

iii. Using definition of determinant, prove that
$\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21}$

iv. Solve the following system of linear equations using Cramer's rule
$x + z = 9, x - 3y = 1, 4y - 3z = 3$

Q4. a) Attempt any **ONE** question from the following: (08)

i. Let V be an inner product space. Define orthogonal vectors. If $\{v_i\}_{i=1}^{n}$ is a set of pair wise orthogonal vectors in V, then show $\|\sum_{i=1}^{n} v_i\|^2 = \sum_{i=1}^{n} \|v_i\|^2$. Is the converse true? Justify your answer.

ii. State and prove Cauchy-Schwarz inequality in an inner product space $(V, \langle\ ,\ \rangle)$. Verify the same for $u = (1,2), v = (2,3)$ from $\mathbb{R}^2$ with Euclidean inner product.

Q4. b) Attempt any **TWO** questions from the following: (12)

i. Let V be a real inner product space and u be an unit vector in V. If $P_u(x)$ denotes the projection of x along u, show that
$\|x - P_u(x)\| \leq \|x - \alpha u\| \ \forall \alpha \in \mathbb{R}$.

ii. Show that $\langle z, w\rangle = Re(z\,\overline{w})$ is an inner product on $\mathbb{C}$ the space of Complex numbers.

iii. Let V be a finite dimensional inner product space over $\mathbb{R}$ and W be a subspace of V. Define $W^{\perp}$, the orthogonal complement of W and prove that $(W^{\perp})^{\perp} = W$.

iv. Define angle between two vectors in an inner product space. Find angle between $p(x) = x^2 + 1$ and $q(x) = x$ using inner product
$\langle p, q\rangle = p(0)q(0) + p(1)q(1) + p(2)q(2)$ where
$p(x) = a_0 + a_1x + a_2x^2$ and $q(x) = b_0 + b_1x + b_2x^2$.

Q5. Attempt any **FOUR** questions from the following: (20)

$a)$ Prove that inverse of Linear Transformation (if it exist) is also a Linear Transformation.

$b)$ Find the rank of $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 3 & 1 & 1 \\ 2 & -2 & 3 \end{pmatrix}$.

$c)$ I) Use determinant to check whether the set $\{(2,0,0,0), (1,-1,0,0), (1,2,5,0), (1,1,1,1)\}$ is linearly dependent or independent. State the result used.

II) Use determinant to find area of the parallelogram spanned by vectors, $x = (1,1), y = (2,5)$. State the result used.

$d)$ Use the following expression of determinant $\det A = \sum_{\sigma \epsilon S_n} sgn\sigma \;\; a_{1\,\sigma(1)}\, a_{2\,\sigma(2)} \ldots\ldots a_{n\,\sigma(n)}$ to find the determinant of the matrix $\begin{pmatrix} 1 & 0 & 3 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{pmatrix}$.

$e)$ Find vectors $u, v \in \mathbb{R}^2$ with Euclidean inner product such that u is a scalar multiple of (1,3); v is orthogonal to (1,3); and $u + v =$(1,2)

$f)$ Find an orthogonal basis of $W = \{(x, y, z) \in \mathbb{R}^3 / x - 2y = z\} \subseteq \mathbb{R}^3$ with dot product using Gram-Schmidt Orthogonalisation Process.

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