(3 Hours) **[Total Marks : 100**

Q.1 Choose correct alternative in each of the following: (20)

i. Let T: V →V' be a linear transformation, then ker(T) and Im(T) are subspaces of

(a) V and V' respectively (b) V' and V respectively

(c) V (d) V'

ii. Let $T:\mathbb{R}^2\to \mathbb{R}^2$ be the linear transformation given by T(x ,y)=(x+ y, y), then Im(T) is equal to

(a) L{(1,0),(0,1)} (b) L{(1,0),(1,1)}

(c) L{(0,1),(0,1)} (d) None of the above

iii. Let $T:\mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by T(x,y,z) = (0,y,z), then KerT is the

(a) X –axis (b) Y-axis

(c) Z -axis (d) None of the above.

iv. If $\mathbf{E} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$ then $\mathbf{E}^{-1} =$

(a) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$ (d) None of the above

v. Which of the following vectors are collinear?

(a) $(2,3), (4,4)$ (b) $(2,3), (4,6)$

(c) $(2,3), (4,6), (1,2)$ (d) $(2,3), (4,6), (1,2), (2,2)$

vi. Let $\{e_1, e_2, e_3\}$ be the standard basis of $\mathbb{R}^3$, then $Det(2e_{2,}, e_1+5e_2, -e_3) =$

(a) -10 (b) -2

(c) $+2$ (d) $+10$

vii. Volume of parallelepiped spanned by vectors $(1,2,3), (4,0,1)$ and $(6,1,1)$ is

(a) 5 (b) 10

(c) 15 (d) None of the above

viii. If G is a group such that $a^3 = e, \forall a \in G$, then

(a) G must be abelian (b) G must be non-abelian

(c) $order(a) = 3, \forall a \in G$ (d) None of the above

ix. The order of the group S_n is

(a) n (b) $n!$

(c) $\phi(n)$ (d) None of the above

x. Which of the following is true?

(a) D_3 and D_4 are abelian groups. (b) D_4 is an abelian group.

(c) D_3 is an abelian group. (d) None of the above.

Q.2 a) Attempt any ONE question from the following: (08)

i. State and prove the Rank-Nullity theorem.

ii. Prove that any n dimensional real vector space is isomorphic to $\mathbb{R}^n$.

b) Attempt any TWO questions from the following: (12)

i. Check whether the linear transformation $T:\mathbb{R}^2 \to \mathbb{R}^2$ given by $T(x, y) = (x + y, y)$ is one-one.

ii. Find the matrix of the linear transformation $T:\mathbb{R}^3 \to \mathbb{R}^2$ given by $T(x, y, z) = (x, 2y + z)$ with respect to the standard basis.

iii. Find ker(T) and Im(T) for the linear transformation $T: \mathbb{R}^2 \to: \mathbb{R}^2$ given by $T(x, y) = (x + y, y)$.

iv. Let $T_1, T_2: V \to V'$ be two linear transformations then prove that $T_1+T_2: V \to V'$ is also a linear transformation.

Q.3 a) Attempt any ONE question from the following: (08)

i. Prove that $\mathbf{S = x_0 + S_h}$ where S is the set of all solutions of the non-homogeneous system of m linear equations in n unknowns $\mathbf{AX = B}$, x_0 is a particular solution of the system and S_h is the set of all solutions of the associated homogeneous system $\mathbf{AX = 0}$.

Hence find the general solution of the system

$$\begin{pmatrix} 2 & 1 & 1 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$ given that $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ is a particular solution of the given non-homogeneous system.

ii. Let $A^1, A^2 \in \mathbb{R}^2$ and $k \in \mathbb{R}$. Show that

p) $\det(A^1, A^2) = 0$ iff $\{A^1, A^2\}$ is linearly dependent.

q) $\det(A^1, A^2 + kA^1) = \det(A^1, A^2)$.

b) Attempt any TWO questions from the following: (12)

i. Express $\mathbf{A} = \begin{pmatrix} 5 & 2 \\ 4 & 2 \end{pmatrix}$ as a product of elementary matrices.

ii. Define rank of a matrix and find rank of $\mathbf{A} = \begin{pmatrix} 2 & 2 & 4 \\ 2 & 2 & 0 \\ 2 & 1 & 4 \end{pmatrix}$

iii. Write the definition of determinant for a 2×2 matrix.
Hence, find the determinant of the matrix $A = \begin{pmatrix} 2 & 3 \\ 5 & 2 \end{pmatrix}$

iv. State the result for inverse of a matrix in terms of its adjoint.
Hence find A^{-1} for $A = \begin{pmatrix} 2 & 3 & 1 \\ 3 & 1 & 2 \\ 1 & 1 & 0 \end{pmatrix}$.

Q.4 a) Attempt any ONE question from the following: (08)

i. Define subgroup. Let G be a group and H, K be subgroups of G. Prove that $H \cup K$ is a subgroup of G iff $H \subseteq K$ or $K \subseteq H$.

ii. Define Group. Prove that the set of prime residue classes modulo n, $U(n)$ is a group under multiplication modulo n.

b) Attempt any TWO questions from the following: (12)

i. Define $GL_2(\mathbb{R})$ and $SL_2(\mathbb{R})$. Prove that $SL_2(\mathbb{R})$ is a subgroup of $GL_2(\mathbb{R})$ under matrix multiplication.

ii. Let G be a group. Prove that $o(a) = o(bab^{-1}), \forall a, b \in G$.

iii. Let Q_8 be the quartenion group. $Q_8 = \{1, -1, i, -i, j, -j, k, -k\}$. Multiplication in Q_8 is defined by $i^2 = j^2 = k^2 = ijk = -1$.
Construct the composition table for Q_8 and show that $order(\pm i) = order(\pm j) = order(\pm k) = 4$.

iv. Show that all subgroups of the additive group $(\mathbb{Z}, +)$ are of the form $n\mathbb{Z}$, where $n \in \mathbb{N} \cup \{0\}$.

Q.5 Attempt any FOUR questions from the following: (20)

a) Let T:U→V and S:V→ W be linear transformations. Prove that SoT is also a linear transformation.

b) If T:V→W is an injective linear transformation and B ={ v_1, v_2, ……..v_n} is linearly independent subset of V then show that { T(v_1), T(v_2) ,………T(v_n)} is linearly independent subset of W.

c) Prove that an elementary 2×2 matrix is invertible.

d) State the Laplace expansion formula for the determinant of an $n \times n$ matrix.
Use Laplace expansion to find the determinant of the matrix

$$A = \begin{pmatrix} 1 & -1 & 3 & 1 \\ -1 & 2 & -1 & 0 \\ 2 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

e) Show that $H = \{I, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$ is a subgroup of S_4.

f) Prove that $G = \left\{ \begin{pmatrix} a & a \\ a & a \end{pmatrix} \middle| a \in \mathbb{R}, a \neq 0 \right\}$ is a group under matrix multiplication.
