(3 Hours) [Total Marks: 100]

Note: (i) All questions are compulsory.
(ii)Figures to the right indicate marks for respective parts.

Q.1 Choose correct alternative in each of the following (20)

$i.$ How many transpositions does S_7 have?

(a) $\frac{7!}{2}$ (b) 14

(c) 21 (d) 42

$ii.$ What is the number of even permutations in S_5?

(a) 5 (b) 60

(b) 120 (d) 1

$iii.$ The recurrence relation for the number of ways to arrange n distinct objects in a row is

(a) $h_n = h_{n-1} + n, h_1 = 1$ (b) $h_n = nh_{n-1}, h_1 = 1$

(c) $h_n = nh_{n-1}, h_1 = 0$ (d) None of these

$iv.$ If A is a countable set and B is an uncountable set, then $A \cup B$ is

(a) Finite (b) Countable

(c) Uncountable (d) None of these

$v.$ A student can choose a computer project from one of three lists. The three lists contain 23, 15 and 19 possible projects, respectively. How many possible projects are there to choose from?

(a) 38 (b) 57

(c) 34 (d) None of these

$vi.$ Which of the following is a partition of {1, 2, 3, 4, 5, 6}?

(a) $\{\{1,2\},\{2,3,4,5,6\}\}$ (b) $\{\{1,2,3\},\{4,5\}\}$

(c) $\{\{1\},\{2,3,4,5,6\}\}$ (d) None of these *vii.* A basket of fruit is being arranged out of apples, bananas, and oranges. What is the smallest number of pieces of fruit that should be put in the basket in order to guarantee that either there are at least 8 apples or at least 6 bananas or at least 9 oranges?

(a) 12 (b) 21

(c) 20 (d) None of these

viii. $\sum_{i=0}^{n} \binom{n}{i} =$

(a) 2^n (c) 1

(c) n (d) None of the above

ix. How many positive integers not exceeding 1000 are divisible by 7 or 11?

(a) 232 (c) 220

(c) 244 (d) None of the above

x. If n prime, n > 1, then $\phi(n)$ is:

(a) $n-1$ (c) n

(c) $n+1$ (d) None of the above

Q2. Attempt any **ONE** question from the following: (08)

a) i. Prove that any cycle can be expressed as a product of transpositions.

ii. The Tower of Hanoi consists of three pegs mounted on a board together with disks of different sizes. Initially these disks are placed on the first peg in order of size, with the largest at the bottom. The rules of the puzzle allow disks to be moved one at a time from one peg to another as long as a disk is never placed on top of a smaller disk. The goal of the puzzle is to have all the disks on the second peg in order of size, with the largest at the bottom. Let H_n denote the minimum number of moves needed to solve the Tower of Hanoi problem with n disks. Set up a recurrence relation for the sequence $\{H_n\}$ and solve it by back-tracking.

Q.2 Attempt any **TWO** questions from the following: (12)

b) i. For the permutation $\sigma = (1\ 3\ 5\ 2\ 4)$

(I) Express σ in two row notation. (II) Find the inverse of σ.

(III) Express σ as a product of transposition and find the sign of σ.

ii. (I) Find the composite of $\sigma = (1\ 3)(1\ 2)(1\ 2\ 3)$ in S_3 in one row notation.
(II) Show that $(2\ 4)(1\ 3) = (1\ \ 3)(2\ 4)$ in S_4.

iii. Find a recurrence relation for h_n, the number of n-digit ternary sequences without any occurrence of the subsequence '012'.

iv. Solve the linear homogeneous recurrence relation
$h_n - 4h_{n-1} + 4h_{n-2} = 0, n \geq 2, h_0 = 1, h_1 = 6$
by using characteristic equation.

Q3. Attempt any **ONE** question from the following: (08)

a) i. Define countable set and give an example of the same. Also show that the set of all integers is countable.

ii. State the recurrence relation for $S(n, k)$ and find the value of $S(6, 3)$. State the value of $S(n, n)$ and $S(n, 1)$.

Q3. Attempt any **TWO** questions from the following: (12)

b) i. Show that $[0, 1] \sim (0, 1)$

ii. Let $A = \{a, b, c, d\}$. Find Stirling number of second kind for $k = 1, 2, 3, 4$ by actually partitioning of A into k parts.

iii. Prove by mathematical induction, $S(n, 2) = 2^{n-1} - 1,\ n \geq 2$

iv. State Pigeonhole principle. In Algebra class, 32 of the students are boys. Each boy knows five of the girls in the class and each girl knows eight of the boys. How many girls are in the class?

Q4. Attempt any **ONE** question from the following: (08)

a) i. Let S be an n-set and suppose the n objects in S are to be put in r distinct boxes $B_1, B_2, \ldots, B_r$ such that the i^{th} box B_i contains n_i objects with $n_1 + n_2 + \cdots + n_r = n$. Then show that the number of ways of doing this is equal to $\binom{n!}{n_1!, n_2!, \ldots, n_r!} = \frac{n!}{n_1!\, n_2! \ldots n_r!}$.

ii. For $n \geq 1$, prove that the number of derangements D_n on n symbols is equal to $D_n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}\right)$.

Q4. Attempt any **TWO** questions from the following: (12)

$b)$ i. There are 15 people enrolled in a mathematics course, but exactly 12 attend on any given day. There are 25 seats in the classroom. Find the number of different ways in which an instructor might see the 12 students in the classroom.

ii. Find the number of ways in which 5 persons A, B, C, D and E can be seated at a round table, such that: (i) A and B must always sit together and (ii) C and D must not sit together.

iii. How many arrangements or digits $0, 1, 2, \ldots, 9$ contains atleast one of the patterns 145, 123, 354?

iv. Determine the total number of integral solutions of $x_1 + x_2 + x_3 = 10$ with $x_1 \geq -1,\ x_2 \geq 2$ and $x_3 \geq 3$.

Q5. Attempt any **FOUR** questions from the following: (20)

$a)$ Show that two cycles given by the same permutation are either identical or disjoint.

$b)$ Solve the linear non-homogeneous recurrence relations $h_n = 3h_{n-1} - 4n, h_0 = 2$.

$c)$ Given eight different English books, seven different French books, and five different German books:

(a) How many ways are there to select one book?

(b) How many ways are there to select three books, one of each language?

$d)$ State strong form of Pigeonhole Principle (Extended Pigeonhole Principle). Given 5 points in the plane with integer coordinates, show that there exists a pair of points whose midpoint also has integer coordinates.

$e)$ What is the number of nondecreasing sequences of length r whose terms are taken from $1, 2, \ldots, k$? Explain your answer.

$f)$ Define Euler ϕ function. Hence find $\phi(16848)$.

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