Time: 2:30 Hours Total Marks: 75

N.B: 1) All questions are compulsory
2) From questions 1, 2 and 3 attempt any **One** from part (a) and any **Two** from part (b)
3) Attempt any **Three** from question 4
4) Figures to the right indicate marks

1\. (a) i) Let $f:[a,b] \to \mathbb{R}$ be a bounded function. If $P,\ Q$ are partitions of $[a,b]$, then Prove that 1) $L(P,f) \le U(P,f)$ 2) $L(P,f) \le U(Q,f)$ 8

ii) Let $f,g:[a,b] \to \mathbb{R}$ be Riemann integrable on $[a,b]$. Prove that $f+g$ is Riemann integrable on $[a,b]$ and hence show that $\int_a^b (f+g) = \int_a^b f + \int_a^b g$ 8

(b) i) Let $f:[a,b] \to \mathbb{R}$ be a continuous function on $[a,b]$. Show that f is Riemann integrable on $[a,b]$. 6

ii) Let f(x) = $\begin{cases} 1-x & if\ 0 \le x < \frac{1}{2} \\ \frac{x}{2} & if\ \frac{1}{2} \le x \le 1 \end{cases}$ 6

$F:[0,1] \to \mathbb{R}$ is defined by $F(x) = \int_0^x f(t)dt,\ x \in [0,1]$. Show that F is differentiable at ½ and $F'(1/2) = f(1/2)$.

iii) Let $f:[0,1] \to \mathbb{R}$ be defined by $f(x) = x^2$. Using Riemann criterion show that f is Riemann integrable on $[0,1]$. 6

iv) Express the sum $\sum_{r=0}^{n-1} \frac{1}{\sqrt{n^2-r^2}}$ as a Riemann sum of a suitable function and evaluate $\lim_{n\to\infty} \sum_{r=0}^{n-1} \frac{1}{\sqrt{n^2-r^2}}$ 6

2\. (a) i) State and Prove Fubini's theorem for a rectangular domain in $\mathbb{R}^2$. 8

ii) Define double integral of a bounded function $f:Q \to \mathbb{R}$ where $Q = [a,b]\ x\ [c,d]$ is a rectangle in $\mathbb{R}^2$. Further show with usual notations 8
$m(b-a)(d-c) \le \iint_Q f \le M(b-a)(d-c)$

(b) i) State the change of variable formula for triple integrals. Stating clearly the conditions under which it is valid. Express further, how will you use it to express the triple integral in Spherical coordinates. 6

ii) Evaluate $\int_0^1 \int_{y^2}^{y} x\,dxdy$ by reversing the order of integration. Sketch the region of integration. 6

iii) Introduce suitable change of variables and show that 6
$\iint_S f(xy)dxdy = \ln 2 \int_1^2 f(u)du$, where S is the region in the 1st quadrant bounded by the curves $xy = 1,\ xy = 2,\ y = x,\ y = 4x$.

iv) Find the volume of the solid S by using triple integration where S is bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 2$ 6

3 (a) i) Let $\{f_n\}$ be a sequence of real valued R-integrable functions on $[a, b]$. If $\{f_n\}$ converges uniformly to f on $[a, b]$ then show that f is R-integrable on $[a, b]$ and $\int_a^b \lim_{n\to\infty} f_n(x)dx = \lim_{n\to\infty} \int_a^b f_n(x)dx$ 8

ii) Let $\{f_n\}$ be a sequence of continuously differentiable real valued functions defined on $[a, b]$. If the series $\sum_{n=1}^{\infty} f_n$ converges pointwise to f on $[a, b]$ and the series $\sum_{n=1}^{\infty} f_n{}'$ converges uniformly on $[a, b]$, then show that $f'(x) = \sum_{n=1}^{\infty} f_n'(x)$ for $a \le x \le b$ 8

(b) (i) Let $\{f_n\}$ be a sequence of real values functions defined on a non-empty subset S of $\mathbb{R}$. Show that $\{f_n\}$ converges uniformly to a function f if and only if for given $\epsilon > 0$, $\exists$ a positive integer n_o such that $|f_n(x) - f_m(x)| < \epsilon$ for $m, n \ge n_o$ and each $x \in S$ 6

(ii) Discuss the pointwise and uniform convergence of the series of functions $\sum_{n=1}^{\infty} \frac{1}{(nx)^2}$, x≠ 0 6

(iii) By integrating a suitable power series term by term show that $\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$ for $|x| < 1$ 6

(iv) Let $f_n: [0, 1] \to \mathbb{R}$ be defined by $f_n(x) = \begin{cases} n(1-nx) & for\ 0 < x < \frac{1}{n} \\ 0 & otherwise \end{cases}$ 6

Check whether $\int_0^1 \lim_{n\to\infty} f_n(x)dx = \lim_{n\to\infty} \int_0^1 f_n(x)dx$. Does $\{f_n\} \to f$ uniformly? Justify.

4 i) Prove that if f:[a,b] → ℝ is Riemann integrable then |f| is Riemann integrable on [a, b]. Is converse true? Justify 5

ii) If $f, g: [a, b] \to \mathbb{R}$ are Riemann integrable and have antiderivatives F and G on $[a, b]$ then show that $\int_a^b F(x)g(x) = [f(b)G(b) - F(a)G(a)] - \int_a^b f(x)G(x)$ 5

iii) Evaluate the following integral by using polar coordinates 5

$\int_0^{\sqrt{2}} \int_y^{\sqrt{4-y^2}} \frac{dxdy}{1+x^2+y^2}$

iv) Use spherical coordinates evaluate $\iiint_S xe^{(x^2+y^2+z^2)^2}\, dxdydz$ where S is the solid that lies between the spheres $x^2 + y^2 + z^2 = 1\ and\ x^2 + y^2 + z^2 = 4$ 5

v) If a real power series $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence r, then show that it converges uniformly on [-s, s] where $0 \le s < r$. 5

vi) Let $f_n(x) = \frac{x^n}{1+x^n}$ for $0 \le x \le 1$. Discuss the pointwise and uniform convergence of $\{f_n\}$ on $[0, 1]$. 5

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