Time: 2:30 Hours Total Marks: 75

N.B: 1) All questions are compulsory
2) From questions 1, 2 and 3 attempt any **One** from part (a) and any **Two** from part (b)
3) Attempt any **Three** from question 4
4) Figures to the right indicate marks

1. (a) i) Let $f: [a, b] \to \mathbb{R}$ be a bounded function. Show that f is Riemann integrable on $[a, b]$ if and only if for each $\epsilon > 0$, there is a partition P_ϵ of $[a, b]$ such that $U(P_\epsilon, f) - L(P_\epsilon, f) < \epsilon$ 8

ii) Let $f: [a, b] \to \mathbb{R}$ be a bounded function and $a < c < b$. Prove that f is Riemann integrable on $[a, b]$ if and only if f is Riemann integrable on $[a, c]$ and $[c, b]$ and further show that $\int_a^b f = \int_a^c f + \int_c^b f$ 8

(b) i) Let$f: [a, b] \to \mathbb{R}$ be a monotone function. Show that f is Riemann integrable on $[a, b]$ 6

ii) Show that a function f:[0,3] $\to \mathbb{R}$ defined by $f(x) = \lfloor x \rfloor$ (floor x) is Riemann integrable on [0, 3] 6

iii) Let $f: [0,1] \to \mathbb{R}$ be defined by $f(x) = x^2$. Let $\{ P_n \}$ be a sequence of partitions $\left\{0, \frac{1}{n}, \frac{2}{n}, \ldots\ldots, \frac{n-1}{n}, 1\right\}$ of $[0, 1]$. Evaluate $\lim_{n\to\infty}(U(P_n, f) - L((P_n, f))$. Also find $\int_0^1 f(x)dx$ 6

iv) Express the sum $\sum_{r=0}^{n-1} \frac{1}{\sqrt{n^2-r^2}}$ as a Riemann sum of a suitable function and evaluate $\lim_{n\to\infty} \sum_{r=0}^{n-1} \frac{1}{\sqrt{n^2-r^2}}$ 6

2. (a) i) State and Prove Fubini's theorem for a rectangular domain in $\mathbb{R}^2$. 8

ii) Define double integral of a bounded function $f: Q \to \mathbb{R}$ where $Q = [a, b] \, x \, [c, d]$ is a rectangle in $\mathbb{R}^2$. Further show with usual notations $m(b-a)(d-c) \le \iint_Q f \le M(b-a)(d-c)$ 8

(b) i) State the change of variables formula for double integral clearly stating the conditions under which it is valid. Explain further how will you use it to express the double integral in polar coordinates. 6

ii) Evaluate $\int_0^9 \int_{\sqrt{x}}^3 \sqrt{1+y^3}\, dydx$ by reversing the order of integration. Sketch the region of integration. 6

iii) Evaluate $\iint_D (x+2y)dxdy$ where D is the region bounded by the straight lines $y = x+3$, $y = x-3$, $y = -2x+4$, $y = -2x+2$ making a suitable change of variables 6

iv) Find the volume of the solid S by using double integration where S is bounded by the coordinate planes and a plane x + y + z = 1 6

3 (a) i) Let $\{f_n\}$ be a sequence of continuous real valued functions defined on a non-empty subset S of $\mathbb{R}$ .If $\{f_n\}$ converges uniformly to a function f on S then show that f is continuous on S. Further show that $\lim_{n\to\infty} \lim_{x\to p} f_n(x) = \lim_{x\to p} \lim_{n\to\infty} f_n(x)$ for each $p \in S$. 8

ii) Let $\{f_n\}$ be a sequence of continuously differentiable real valued functions defined on $[a,b]$. If the series $\sum_{n=1}^{\infty} f_n$ converges pointwise to f on $[a,b]$ and the series $\sum_{n=1}^{\infty} f_n{}'$ converges uniformly on $[a,b]$, then show that $f'(x) = \sum_{n=1}^{\infty} f_n'(x)$ for $a \le x \le b$ 8

(b) (i) State and prove Weierstrass M test for uniform convergence of series of functions. 6

(ii) Show that the sequence of functions $f_n : [0,\infty) \to \mathbb{R}$ defined by $f_n(x) = \frac{x}{n} e^{\frac{-x}{n}}$ does not converge uniformly on $[0,\infty)$ but converge uniformly on $[0,a]$; $a > 0$ 6

(iii) By differentiating the power series for $\frac{1}{1-x}$ term by term in $[-a,a]$ $0 \le a < 1$. Show that $\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \ldots\ldots\ldots$ 6

(iv) Show that the series $\sum_{n=0}^{\infty} \frac{(-1)^n n + x^n}{n^2}$ converges uniformly on $[-1,1]$ 6

4 i) Let $f: [0,1] \to \mathbb{R}$ be defined by $f(x) = \begin{cases} 0 & if\ x = 0\ or\ x\ is\ irrational \\ \sin n\pi\ \ if\ x = \frac{m}{n}; & m,n \in \mathbb{N}\ and\ \ co-prime \end{cases}$ 5

Is f Riemann integrable on $[0,1]$? Justify.

ii) Prove that if f is a continuous function on a closed bounded interval $[a,b]$ then there exists at least one c in $[a,b]$ such that $f(c)(b-a) = \int_a^b f(t)dt.$ 5

iii) Evaluate the following integral by using polar coordinates 5

$$\int_{-3}^{3} \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} \sqrt{9 - x^2 - y^2}\, dxdy$$

iv) Use spherical coordinates evaluate $\iiint_S \ xe^{(x^2+y^2+z^2)^2}\ dxdydz$ where S is the solid that lies between the spheres $x^2 + y^2 + z^2 = 1\ and\ x^2 + y^2 + z^2 = 4$ 5

v) If a real power series $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence r, then show that it converges uniformly on [-s, s] where $0 \le s < r$. 5

vi) Discuss the pointwise and uniform convergence of the sequence of functions $\{x^2 e^{-nx}\}$ on $[0, \infty)$ 5