QP CODE : 28878

Duration:[3 Hours] [Marks: 100]

N.B. 1) All questions carry equal marks.
2) Attempt any **five** questions.

1. (a) Let V be an inner product space. State and prove following properties for norm function $\|\cdot\| : V \longrightarrow \mathbb{R}$
   (i) Triangle inequality (3)
   (ii) Cauchy-Schwarz inequality (4)
   (iii) Pythagoras theorem (3)
   (b) (i) Define Special orthogonal group SO_n and rotation of $\mathbb{R}^3$. (2)
   (ii) Show that a matrix A represents a rotation of $\mathbb{R}^3$ if and only if $A \in SO_3$. (8)

2. (a) (i) Show that an isometry which fixes the origin is a linear operator. (5)
   (ii) Give an example of isometry which do not fix the origin. Justify. (5)
   (b) Let A be a $n \times n$ real matrix. Show that the following conditions are equivalent (10)
   (i) A is orthogonal.
   (ii) Multiplication by A preserves dot product.
   (iii) The columns of A are mutually orthogonal unit vectors.

3. (a) Consider initial value problem $\dfrac{dy}{dx} = f(x, y)$ with $y(x_0) = y_0$.
   (i) State Picard's theorem for the existence and uniqueness of solution of initial value problem. (2)
   (ii) Explain Picard's scheme of approximation for the solution of initial value problem. (6)
   (iii) Explain why Picard's theorem known for local existence and uniqueness of solution of initial value problem. (2)
   (b) Find approximate solution upto t^4 of the initial value problem (10)
   $\dfrac{dx}{dt} = 2x + ty$, $\dfrac{dx}{dt} = xy$ with $x(0) = 1$ and $y(0) = 1$.

4. (a) Let S be a regular surface and $p \in S$. Prove that there exists a neighborhood V of p in S such that V is the graph of a differentiable function which has the form $z = f(x, y)$ or $y = g(x, z)$ or $x = h(y, z)$. (10)
   (b) (i) Let $\sigma : U \to \mathbb{R}^3$ be a patch of a surface S containing a point p of S and (u, v) be co-ordinates in U. Then show that the tangent space to S at p is the vector subspace of $\mathbb{R}^3$ spanned by the vectors σ_u and σ_v. (5)
   (ii) Prove or disprove: Every plane in $\mathbb{R}^3$ is a regular surface. (5)

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5. (a) State and prove the fundamental theorem for plane curve. (10)
   (b) The parametric equation of a circular helix with the $z-$ axis is given by

$$\gamma(\theta) = (a\cos\theta, a\sin\theta, b\theta), -\infty < \theta < \infty$$

   (i) Find the curvature of a circular helix. (5)
   (ii) Find the torsion of a circular helix. (5)

6. (a) State and prove the generalized Stoke's theorem for the integration of exterior forms. (10)
   (b) (i) Prove that the local maxima and local minima of function f are critical points of f. (5)
   (ii) Let $f(x, y, z) = (x + y + z - 1)^2$. Locate the critical points and critical values of f. (5)

7. (a) Prove or disprove
   (i) The differential $dN_p : T_p(S) \longrightarrow T_p(S)$ of the Gauss map is a self adjoint linear map. (5)
   (ii) The value of the second fundamental form for a unit vector $v \in T_p(S)$ is equal to the normal curvature of a regular curve passing through p and tangent to v. (5)
   (b) Consider surface covered by the parametrization $\sigma(u, v) = (u + v, u - v, uv)$. Calculate following at the point $(2, 0, 1)$.
   (i) The coefficients of the first fundamental form (2)
   (ii) The coefficients of the Second fundamental form (2)
   (iii) The Gaussian curvature (2)
   (iv) The Principal curvatures (2)
   (v) The Mean curvature (2)

8. (a) Let $\sigma(u, v) = (ucosv, usinv, u)$ where $u = e^{\lambda t}$, $v = t$ and λ is constant. Find the length of part of the curve with $0 \leq t \leq \pi$. (5)
   (b) Compute the tangent, normal and binormal to the curve $\gamma(t) = (\frac{1}{3}(1+t)^{\frac{3}{2}}, \frac{1}{3}(1-t)^{\frac{3}{2}}, \frac{t}{\sqrt{2}})$. (5)
   (c) The parametric equation of Möbius band is $\sigma(t, \theta) = ((1 - tsin\frac{\theta}{2})cos\theta, (1 - tsin\frac{\theta}{2})sin\theta, tcos\frac{\theta}{2})$ Is the Möbius band orientable? Justify. (5)
   (d) Show that the angle of intersection of two curves γ and $\bar{\gamma}$ on a surface S is given by $cos\theta = \frac{F}{EG}$ where E,F and G are notations as in first fundamental form. (5)

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