(3 Hours) [Total Marks: 100]

N.B.: (1) Solve any Five questions from question number 1 to 8.
(2) Figures to the right indicate full marks.

1. (a) State and prove i) Hölder's inequality and ii) Minkowski's inequality. (10)

   (b) Verify that $||x||_1 := |\xi_1| + |\xi_2| + \cdots + |\xi_n|$ $(x = (\xi_1, \xi_2, \cdots, \xi_n) \in \mathbb{C}^n)$ is a norm function on $\mathbb{C}^n$. (10)

2. (a) Prove that any two norms on a finite dimensional vector space over $\mathbb{R}$ are equivalent. (10)

   (b) Show that $\mathscr{C}[a,b]$, the set of all continuous real valued functions on interval $[a,b]$ is complete. (10)

3. (a) State and prove the lemma of Riesz. (10)

   (b) Prove that any linear operator on a finite dimensional normed linear space X is bounded linear operator. (10)

4. (a) Let X, Y are normed linear spaces over $\mathbb{R}$. If Y is a Banach space over $\mathbb{R}$, then prove that $B(X,Y)$ is a Banach space over $\mathbb{R}$. (10)

   (b) If B, B' are Banach spaces over $\mathbb{R}$ and if T is continuous linear operator from B onto B', then prove that the image of the open sphere centered at the origin in B contains an open ball centered at the origin in B'. (10)

5. (a) Let (T_n) be a sequence of bounded linear operators $T_n : X \to Y$ from a Banach space X into a normed linear space Y. Show that if sequence $(||T_n(x)||)$ is bounded for every $x \in X$ then sequence $(||T_n||)$ is bounded. (10)

   (b) Let X be a normed linear space and let $x_0 \neq 0$ be any element of X. Prove that there exists a bounded linear functional f on X such that $||f|| = 1$ and $f(x_0) = ||x_0||$. (10)

6. (a) Prove that l^2 is a Hilbert space. (10)

   (b) State and prove the Cauchy-Schwarz inequality. (10)

7. (a) Let X be a normed linear space over $\mathbb{R}$. Prove that the set of eigenvalues of a compact linear operator $T : X \to X$ on a normed space X is countable, and the only possible point of accumulation is $\lambda = 0$. (10)

   (b) Let $T : X \to X$ be a compact linear operator and $S : X \to X$ be a bounded linear operator on a normed linear space X over $\mathbb{R}$. Then prove that composite $T \circ S$ is a compact operator. (10)

8. (a) Define a compact operator. Let X, Y be normed linear spaces over $\mathbb{R}$ and $F : X \to Y$ be a linear map. Prove that F is a compact operator if and only if for every bounded sequence (x_n) in X, sequence $(F(x_n))$ contains a subsequence which converges in Y. (10)

   (b) Let X, Y be normed linear spaces over $\mathbb{R}$. If $F \in B(X,Y)$ and has finite rank, then prove that the range $R(F)$ is closed in Y and F is compact. (10)

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