(3 Hours) **[Total Marks: 100**

N.B.: (1) Attempt any **FIVE** questions.

(2) Figures to the right indicate marks for respective sub-questions.

1. (a) Show that there is a non-measurable subset in $\mathbb{R}$. (10)

(b) (i) Show that exterior measure of any countable subset of $\mathbb{R}$ is zero. Show by an example that the converse is not true. (5)

(ii) If E_1 and E_2 are Lebesgue measurable subsets of $\mathbb{R}$, then show that $E_1 \cup E_2$ is Lebesgue measurable. (5)

2. (a) If f is a measurable function, then show that

(i) f^2 is measurable (5)

(ii) $\lambda + f$ is measurable, where $\lambda \in \mathbb{R}$. (5)

(b) (i) Let f be a bounded function defined on the closed and bounded interval $[a, b]$. If f is Riemann integrable over $[a, b]$, then show that it is Lebesgue integrable over $[a, b]$ and the two integrals are equal. (5)

(ii) Show by an example that a Lebesgue integrable function may not be Riemann integrable. (5)

3. (a) State and prove Fatou's lemma. Show by an example that the inequality in Fatou's Lemma may be a strict inequality. (10)

(b) (i) Let $\{f_n\}$ be an increasing sequence of non-negative measurable functions on E. If $f_n \to f$ pointwise a.e. on E, then show that $\lim\limits_{n\to\infty} \int_E f_n = \int_E f$. (5)

(ii) Let $E_1 \subseteq E_2 \subseteq \cdots$ be measurable subsets of $\mathbb{R}$ with $E = \cup_{n=1}^{\infty} E_n$. Show that $m(E) = \lim\limits_{n\to\infty} m(E_n)$. (5)

4. (a) State Fubini's theorem. Use Fubini's theorem to evaluate $\int_A (x \sin y - y e^x)\,dxdy$, where $A = [-1, 1] \times [0, \pi/2]$. (10)

(b) (i) Let A be a subset of $\mathbb{R}$. Show that the characteristic function χ_A is measurable if and only if the set A is measurable. (5)

(ii) Let f and g be non-negative integrable functions on a measurable subset E of $\mathbb{R}$. Show that if $f = g$ a.e. then $\int_E f = \int_E g$. (5)

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5. (a) Let (f_n) be a sequence of measurable functions. Show that $\sup_n\{f_n\}$ and $\lim_{n\to\infty} f_n$ are measurable functions. (10)

   (b) Let f and g be two non-negative measurable functions on a measaurable set E and c be a non-negative real number. Show that (10)

   (i) $\int_E (cf) = c\int_E f$

   (ii) $\int_E (f+g) = \int_E f + \int_E g$

6. (a) Show that any separable Hilbert space has a complete orthonormal basis. (10)

   (b) State and prove Holder's inequality. (5)

   (c) State and prove Parseval's identity (5)

7. (a) Show that $\ell^2(\mathbb{N})$ is a complete metric space. (10)

   (b) Let $\{e_n\}_{n\in\mathbb{N}}$ be an arbitrary orthonormal set in $L^2[-\pi,\pi]$ and let $c_1, c_2, \ldots$ be complex numbers such that the series $\sum_{k=1}^{\infty} c_k$ converges. Show that there exist a function $f \in L^2[-\pi,\pi]$ such that $c_k = \langle f, e_k\rangle$ and $\sum_{k=1}^{\infty} c_k^2 = \|f\|^2$ (10)

8. (a) Let f be an integrable function on the circle which is differentiable at a point x_0. Show that $S_N(f)(x_0) \to f(x_0)$ as $N \to \infty$, where $S_N f(x)$ is the N-th partial sum of the Fourier series of f. (10)

   (b) (i) Find the solution of the Dirichlet's problem $\Delta u = 0$ on the unit disc, with the boundary condition $u(1,\theta) = \cos^2\theta$. (5)

   (ii) Find the Fourier series of the function $f(x) = |x|$ in $-\pi \le x \le \pi$. (5)