Time: 3 Hours [Total Marks: 100]

N.B. 1) Attempt any **five** questions out of **eight**.
2) All questions carry equal marks.

Q. 1. (a) State and prove the orbit-stabilizer formula. **(10)**

(b) State without proof any one the Sylow theorems. Explain clearly all the notation used. Determine all the Sylow 2-subgroups of S_3. Show that they are all conjugate. **(10)**

Q. 2. (a) State and prove Jordan Hölder theorem for finite groups. **(10)**

(b) (i) Let G be a group and let H be a normal subgroup of G. Prove that if both H and G/H are solvable, then so is G. **(5)**

(ii) Prove that every group of order 9 is abelian. **(5)**

Q. 3. (a) State and prove Hilbert basis theorem. **(10)**

(b) (i) Define the term: Noetherian ring with one example. If k is a field, is the polynomial ring $k[X_1, X_2, \ldots, X_n, \cdots]$ in countably many variables over k Noetherian? Justify. **(5)**

(ii) State and prove (any one) isomorphism theorems for modules over a commutative ring R with unity. **(5)**

Q. 4. (a) Prove that for every prime p and every natural number n, there exists a finite field of order p^n. **(10)**

(b) (i) Let R be a commutative ring with unity and let M be an R-module. Prove that $\text{Ann } R = \{r \in R \mid rn = 0, \text{ for all } n \in N\}$ is an ideal of R. Further show that if $N \subset L$ are submodules of M, then $\text{Ann}(L) \subset \text{Ann}(N)$. **(5)**

(ii) Let F be a field and X be an indeterminate and let $R = F[X]$. Let V be a vector space over F and T be a linear transformation from V to V. Write a formula for making V into an $F[X]$-module. Verify that V is an $F[X]$-module via this formula. **(5)**

Q. 5. (a) (i) Define the terms: algebraic element and transcendental element. Give one example of each with correct justification. **(5)**

(ii) Determine the degree of the field extension $\mathbb{Q}(\sqrt{5}+\sqrt{7})$ over $\mathbb{Q}$ with correct justification. **(5)**

(b) State and prove primitive element theorem. **(10)**

Q. 6. (a) (i) State (without proof) the fundamental theorem for Galois theory. Explain clearly all the notation used. **(5)**

(ii) Define the term: cyclotomic field. Determine the degree of the cyclotomic field over $\mathbb{Q}$ with correct justification. **(5)**

(b) Determine whether the regular 5-gon is constructible by straightedge and compass. Justify. **(10)**

Q. 7. (a) (i) Prove that the map $a + b\sqrt{2} \mapsto a - b\sqrt{2}$ is an automorphism of $\mathbb{Q}(\sqrt{2})$. Find the fixed field of this automorphism. **(5)**

(ii) Determine the degree of the splitting field of $X^4 - 2$ over $\mathbb{Q}$ with correct justification. **(5)**

(b) Let R be a commutative ring with unity and let M be an R-module. Prove that the following statments are equivalent:

(i) Every non-empty set of submodules of M contains a maximal element under inclusion.

(ii) Every submodule of M is finitely generated.

(10)

Q. 8. (a) (i) Define the terms: Artinian ring, Artinian module. Give one example of each. **(5)**

(ii) Let G be a group acting on a set X. Prove that the stabilizer in G of an element $x \in X$ is a subgroup of G. **(5)**

(b) Let $\sigma = (1\ 2\ 3\ 4\ 5) \in S_5$. Find $\tau \in S_5$ such that $\tau\sigma\tau^{-1} = \sigma^{-1}$. **(10)**