(3 Hours) **[Total Marks: 100**

N.B.: (1) **All questions are compulsory**.
(2) Attempt any **two** subquestions from **part(a)**, **part(b)** and **part(c)**.
(3) Figures to the right indicate marks for respective subquestions.
(4) Use of non-programmable calculator is allowed.

1. (a) If x_{k-1} and x_k are two approximations to the root of $f(x) = 0$, then show that the next approximation x_{k+1} to the root using secant method is

$$x_{k+1} = x_k - \frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})} f(x_k) \quad \text{for} \quad k = 1, 2, \ldots$$

Also show that the rate of convergence of the secant method is $\frac{1}{2}(1 \pm \sqrt{5})$. (10)

(b) Define the shift operator E and central difference operator δ. Show that (10)

(i) $e^x = \left(\frac{\Delta^2}{E}\right) e^x \cdot \frac{Ee^x}{\Delta^2\, e^x}$, where the interval of difference is one.

(ii) $\delta = E^{-1/2}\,\Delta = E^{1/2}\nabla$

(c) Derive the Newton-Cotes Quadrature formula $\int_a^b f(x)dx = \sum_{k=0}^{n} \lambda_k f(x_k)$ and deduce the trapezoidal rule $\int_a^b f(x)dx = \frac{b-a}{2}[f(a) + f(b)]$ using method of interpolation. Hence find the approximate value of $\int_0^1 \frac{dx}{1+x^2}$ using trapezoidal rule. (10)

2. (a) Describe the Newton-Raphson method to find the root of the first degree equation $f(x) = 0$. Further prove that the Newton-Raphson method converge if $|f(x)f''(x)| < |f'(x)|^2$. (10)

(b) Discuss Birge-Vieta method for determining the roots of an algebraic equation. Perform one iteration of the Birge-Vieta method to find the smallest positive root of the equation $2x^2 - 5x + 1 = 0$. (10)

(c) Find all the roots of the polynomial $x^3 - 4x^2 + 5x - 2 = 0$ using the Graeffe's root squaring method. (10)

3. (a) Explain Triangularization method for solving a system of linear equations. Show that the matrix $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \\ -1 & 0 & 2 \end{pmatrix}$ is non-singular but cannot be written as LU product of lower and upper triangular matrices. (10)

(b) Solve the system of equations:

$$\begin{aligned} 10x + y + z &= 12 \\ 2x + 10y + z &= 13 \\ 2x + 2y + 10z &= 14 \end{aligned}$$

using Gauss-Seidal method. Take the initial approximation $x^{(0)} = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}^t$, and perform three iterations. (10)

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(c) Find the largest eigenvalue and the corresponding eigenvector of the matrix $\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$ using power method. Take the initial approximate vector as $v^{(0)} = [1, 1, 1]^T$. (10)

4. (a) With usual notation, show that the Newton's Divided Difference interpolating polynomial $P_n(x)$ for the function $f(x)$ with nodal points $x_0, x_1, \ldots, x_n$ is given by

$$P_n(x) = f[x_0] + (x - x_0)f[x_0, x_1] + \cdots + (x - x_0)\cdots(x - x_{n-1})f[x_0, x_1, \ldots, x_n].$$

Calculate $f(3)$ using above formula for the following data: (10)

x	0	1	2	4	5	6
$f(x)$	1	14	15	5	6	19

(b) Find the unique polynomial $P(x)$ of degree 2 or less such that $P(1) = 1$, $P(3) = 27$ and $P(4) = 64$ using (10)

(i) Lagrange's interpolation formula and

(ii) Iterated interpolation formula.

Estimate $P(1.5)$.

(c) Fit the following four points by the quadratic splines with $f''(0) = 0$. Hence, find an estimate of $f(2.5)$. (10)

x	0	1	2	3
$f(x)$	1	2	33	244

5. (a) Derive the composite Simpson's rule for finding $\int_a^b f(x)dx$ where the interval $[a, b]$ is divided into n equal subintervals with endpoints $[x_i, x_{i+1}]$, $0 \leq i \leq n-1$ where $a = x_0$, $b = x_n$. (10)

(b) Evaluate double integral $\int_1^2 \int_1^2 \frac{dxdy}{x + y}$ using Trapezoidal rule with $h = k = 0.5$. (10)

(c) Derive Gauss-Legendre two point formula $\int_{-1}^1 f(x)dx = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$ using Newton-Cotes Quadrature formula. Using this formula, evaluate the integral $\int_0^2 \frac{dx}{x^2 + 2x + 10}dx$. (10)
