Q. P. Code : 10482

Duration:[$2\frac{1}{2}$ Hours] [Marks: 60]

N.B. 1) All questions are compulsory and carry equal marks.
2) Solve any **Two** from questions 1, 2, 3, 4.
3) Solve any **Four** from 5th question.
4) **All Hilbert spaces are over Complex field.**

1. (a) If f is an integrable periodic function and $\hat{f}(n)$ is its Fourier coefficient then show that
   (i) The Fourier series of f can be written as (3)

$$f(\theta) \sim \hat{f}(0) + \sum_{n\geq 1}[\hat{f}(n) + \hat{f}(-n)]\cos n\theta + i[\hat{f}(n) - \hat{f}(-n)]\sin n\theta.$$

   (ii) $\lim_{|n|\longrightarrow\infty} \hat{f}(n) = 0.$ (3)

   (b) (i) Define the N^{th} Dirichlet's kernel $D_N(x)$ and the N^{th} partial sum $S_N(f)$ of the Fourier series of an integrable periodic function f. (2)
   (ii) State and prove Dirichlet's theorem. (4)

   (c) Prove or disprove: The N^{th} Fejer kernel $F_N(x)$ is a good kernel. (6)

2. (a) (i) Show that every non zero Hilbert space contains a complete orthonormal set. (3)
   (ii) Let $\mathbb{H}$ be a Hilbert space and $\{e_i\}$ be complete orthonormal set in $\mathbb{H}$. If $f \in \mathbb{H}$ is orthogonal to e_i then show that $f = 0$. (3)

   (b) Let $\{e_k\}_{k=1}^{\infty}$ be an orthonormal set in Hilbert space $\mathbb{H}$ then show that for any $f \in \mathbb{H}$, $\sum_{k=1}^{\infty}|\langle f, e_k\rangle|^2 \leq \|f\|^2$. (6)

   (c) Let $\mathcal{C}^N$ denotes finite dimensional complex Euclidean space.
   (i) Define inner product and norm on $\mathcal{C}^N$. (2)
   (ii) Is $\mathcal{C}^N$ Hilbert space? Justify your answer. (4)

3. (a) Show that the Hilbert space $L^2[-\pi, \pi]$ is separable in its metric. (6)

   (b) (i) Define infinite dimensional square summable sequence space of complex numbers $l^2(Z)$. Also define inner product and norm on $l^2(Z)$. (2)
   (ii) Let periodic function $f \in L^2[-\pi, \pi]$ and a_n is Fourier coefficient of f. Is the mapping $f \longmapsto \{a_n\}_{n\in Z}$ a unitary correspondence between $L^2[-\pi, \pi]$ and $l^2(Z)$? Justify your answer (4)

   (c) Consider f is a continuous periodic function in $L^2[-\pi, \pi]$.
   (i) Show that the partial sum of Fourier series of f converges to f in $L^2[-\pi, \pi]$. (3)
   (ii) State and prove Parseval's identity for $f \in L^2[-\pi, \pi]$. (3)

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4. (a) State and prove the Weierstrass approximation theorem. (6)

(b) If f be an integrable function defined on the circle and has a jump discontinuity at θ then show that (6)

$$\lim_{r \longrightarrow 1} A_r(f)(\theta) = \frac{f(\theta^+) + f(\theta^-)}{2}, \; 0 \leq r < 1$$

where $A_r(f)(\theta)$ denotes the Abel mean of Fourier series of function f.

(c) (i) State the Dirichlet's problem in the unit disc. (1)

(ii) Let f be an integrable function defined on the unit circle and $P_r(\theta)$ denotes the Poisson kernel. Show that solution of the Dirichlet's problem in the unit disc is given by the Poisson integral $u(r, \theta) = (f * P_r)(\theta)$. (4)

(iii) Comment on the uniqueness of solution of the Dirichlet's problem. (1)

5. (a) The initial position of the triangular shaped plucked string to the height h is given by (3)

$$f(x) = \begin{cases} \dfrac{xh}{p}, & 0 \leq x \leq p, \\ \dfrac{h(\pi - x)}{\pi - p}, & \text{p} \leq x \leq \pi \end{cases}$$

Show that the Fourier series expansion of f is given by $f(x) = \sum_{m=1}^{\infty} A_m \sin mx$ where $A_m = \dfrac{2h \sin mp}{m^2 p(\pi - p)}$.

(b) Suppose that $f : I\!R \to I\!R$ is continuous periodic function such that (3)

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx = 0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx$$

for all $n \in \; Z^+$ then prove that f is identically zero.

(c) Let f be the function defined on $[-\pi, \pi]$ by $f(\theta) = |\theta|$. Use Parseval's identity to find the sum of series $\sum_{n=0}^{\infty} \dfrac{1}{(2n+1)^4}$. (3)

(d) If f is integrable function defined on the circle then show that (3)

$$\|f - S_N(f)\| \leq \|f - \sum_{|n| \leq N} c_n e_n\|$$

for any complex number c_n, where $S_N(f)$ is the N-th partial sum of Fourier series of f and $\{e_n\}$ be an orthonormal set.

(e) Find a function f_r whose Fourier coefficients are $\hat{f}_r(n) = r^{|n|}$ where $0 \leq r < 1$. (3)

(f) Give an example of a series which Abel summable but not Cesaro summable. Justify your answer. (3)

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