Q.P.Code : 07056

($2\frac{1}{2}$ Hours) OLD COURSE [Total Marks :75]

N.B.: (1) All questions are compulsory.

(2) Figures to the right indicate marks for respective sub questions.

1. (a) Attempt any **one** question:

(i) State and Wilson's Theorem. Is converse true? Justify . (08)

(ii) Let p be a prime. Show that $x^2 \equiv -1(mod\ p)$ has solutions if and only if p=2 or $p \equiv 1(mod\ 4)$. Hence show that the odd prime divisors of the integer n^2 +1 are of the form 4k+1. (08)

(b) Solve any **two** of the following.

(i) Given any integer k > 0, show that there exist k consecutive composite numbers. Show that there are infinitely many primes of the form 4k+3. (06)

(ii) Assume that p and q are distinct odd primes such that (p-1)| (q-1) If gcd (a, pq)=1, show that $a^{q-1} \equiv$ 1(mod pq). (06)

(iii) State and prove Chinese Remainder Theorem (06)

(iv) Prove that if the integer n has r distinct odd prime factors, then $2^r | \emptyset(n)$ where $\emptyset(n)$ is Euler function. (06)

Q2 (A) Attempt any **one** question: (08)

(i) Let the positive integer n be written as n= N^2m, where m is square free. Then show that n can be represented as the sum of two squares if and only if m contains no prime factor of the form 4k+3.

(ii) Prove that the Diophantine equation $x^4 + y^4 = z^2$ has no solution in positive integers x, y, z. (08)

(B) Solve any **two** of the following

(i) If x, y and z is a primitive Pythagorean triple prove that x+y and x-y are congruent modulo 8 either to 1 or 7. (06)

(ii) If n is the sum of two triangular numbers establish that 4n+1 is the sum of two squares. (06)

(iii) Establish that each of the integers 2^n , where n = 1,2,3,……., is a sum of two squares. (06)

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(iv) Solve the linear Diophantine equation 172x + 20y =1000 (06)

3. (a) Attempt any **one** question:

(i) State and prove the Quadratic Reciprocity Law. (08)

(ii) If p is an odd prime and gcd(a,2p)=1 then show that (08)

$\left(\frac{a}{p}\right) = (-1)^t$ where $t=\sum_{j=1}^{\frac{p-1}{2}}\left[\frac{ja}{p}\right]$ and $\left(\frac{2}{p}\right) = (-1)^{\frac{p^2-1}{8}}$

(b) *Solve any **two** of the following.*

(i) Prove that if p is an odd prime and gcd(a,p)=1 then the congruence (06) $x^2 \equiv a(mod p^n)$ n≥ 1 has solution if and only if $\left(\frac{a}{p}\right)$=1.

(ii) Find whether $x^2 \equiv 231\ (mod\ 1105)$ is solvable (06)

(iii) Show that $\left(-\frac{3}{p}\right)$= 1 if p ≡ 1(mod 6) and $\left(-\frac{3}{p}\right)$ =-1 if p≡ 5 (mod 6) (06)

(iv) Define Jacobi symbol $\left(\frac{P}{Q}\right)$ for Q odd and positive.Show that if Q is odd (06) and Q>0 then $\left(\frac{-1}{Q}\right) = (-1)^{\frac{Q-1}{2}}$ and $\left(\frac{2}{Q}\right) = (-1)^{\frac{Q^2-1}{8}}$.

Q4 *Solve any **three** of the following*

(i) Establish each of the following assertions: (05)

a) If n is an odd integer, then $\Phi(2n) = \Phi(n)$

b) If n is an even integer , then $\Phi(2n) = 2\,\Phi(n)$.

(ii) State Fermat's Theorem. Use it to verify that 17 divides 11^{104}+ 1. (05)

(iii) Prove that in a primitive Pythagorean triple x, y, z, the product xy is divisible by 12, hence 60 divides xyz. (05)

(iv) Prove that the radius of the inscribed circle of a Pythagorean triangle is always an integer. (05)

(v) Let p be an odd prime and gcd(a, p)= gcd(b, p) =1. Prove that either all three of the congruences (05)

$x^2 \equiv a(mod\ p)$ $\quad x^2 \equiv b(mod\ p)$ $\quad x^2 \equiv ab(mod\ p)$

are solvable or exactly one of them admits a solution.

(vi) Prove that 2 is not a primitive root of any prime of the form p = 3.2^n + 1 except when p=13. (05)
