Examination : SYBA_Semester IV **Subject : Mathematics (Paper III)**
Exam Date : 02-05-2019 **Q.P.Code : 66043**

(3 Hours) [Total Marks: 100]

Note: (i) All questions are compulsory.
(ii)Figures to the right indicate marks for respective parts.

Q.1 Choose correct alternative in each of the following (20)

$i.$ Let $(G, \cdot)$ be a group such that $\forall\, x, y \in G, (xy)^n =$ then

(a) e (b) $x^n y^n$ if and only if G is abelian

(c) $(xy)^{-1}$ (d) None of the above

Ans **(b) $\boldsymbol{x^n y^n}$ if and only if $\boldsymbol{G}$ is abelian**

$ii.$ The set U_{10} forms a group under multiplication. Then the elements of U_{10} are

(a) $\{\bar{1}, \bar{3}, \bar{7}, \bar{9}\}$ (b) $\{\bar{1}, \bar{2}, \cdots\cdots, \bar{9}\}$

(c) $\{\bar{1}\}$ (d) None of the above

Ans **(a) $\{\bar{\mathbf{1}}, \bar{\mathbf{3}}, \bar{\mathbf{7}}, \bar{\mathbf{9}}\}$**

$iii.$ Let S_4 denote the permutation group on 4 symbols. Then $|S_4| =$

(a) 24 (b) 4

(c) 16 (d) None of the above

Ans **(a) 24**

$iv.$ G is a cyclic group having exactly 3 subgroups namely G itself, $\{e\}$ and a subgroup of order 7. Then order of G is

(a) 14 (b) 49

(c) 7 p where p is any prime (d) cannot say

Ans **(b) 49**

$v.$ Let G be a cyclic group of infinite order. Then the number of elements of finite order in G is

(a) 0 (b) 1

(c) 2 (d) infinite

Ans **(b) 1**

$vi.$ Let $G = \langle a \rangle$ and order of G be 40. Then the elements of order 10 in G are

(a) a^4, a^8, a^{12}, a^{16} (b) $a^4, a^{12}, a^{28}, a^{36}$

(c) $a^4, a^{16}, a^{24}, a^{32}$ (d) a^4, a^8, a^{24}, a^{28}

Ans **(b) $\boldsymbol{a^4, a^{12}, a^{28}, a^{36}}$**

vii. Suppose $a \in G$ such that $o(a) = 20$ then number of cosets of $\langle a^5 \rangle$ in $\langle a \rangle$ is

(a) 3 (b) 4

(c) 5 (d) None of these

Ans **(c) 5**

viii. Let H and K be subgroups of G such that $o(H) = 12$ and $o(K) = 35$ then o($H \cap K$) is

(a) 2 (b) 1

(c) 3 (d) None of these

Ans **(b) 1**

ix. Let G be a cyclic group of order 7 and $\phi: G \longrightarrow G$ given by $\phi(x) = x^4$, then

(a) ϕ is not a group homomorphism.

(b) ϕ is a group homomorphism which is not one $-$one.

(c) ϕ is a group homomorphism which is not onto.

(d) None of these

Ans **(d) None of these**

x. Let $G = D_n, n \in \mathbb{N}$. Define $f: G \longrightarrow \{1, -1\}$ by $f(x) = 1$ if x is a rotation
$= -1$ if x is reflection

then $kerf$ is a subgroup of G containing only

(a) Reflection (b) Rotation and reflection

(c) Rotation (d) None of these

Ans **(c) Rotation**

Q2. Attempt any **ONE** question from the following: (08)

a) i. Define Subgroup. Hence or otherwise show if H and K are the subgroups of a group G. Then $H \cup K$ is a subgroup if and only if $H \subseteq K$ or $K \subseteq H$.

Ans **Solution.** Suppose H_1 and H_2 are two subgroups of a group G. Let $H_1 \subseteq H_2$ or $H_2 \subseteq H_1$. Then $H_1 \cup H_2 = H_2$ or H_1. But H_1, H_2 are subgroups and therefore $H_1 \cup H_2$ is also a subgroup. 2

Conversely suppose $H_1 \cup H_2$ is a subgroup. To prove that $H_1 \subseteq H_2$ or $H_2 \subseteq H_1$.

Let us assume that H_1 is not a subset of H_2 and H_2 is also not a subset of H_1.

Now H_1 is not a subset of $H_2 \Rightarrow \exists\, a \in H_1$ and $a \notin H_2$ ...(1)

and H_2 is not a subset of $H_1 \Rightarrow \exists\, b \in H_2$ and $b \notin H_1$. ...(2) 2

From (1) and (2), we have $a \in H_1 \cup H_2$ and $b \in H_1 \cup H_2$.

Since $H_1 \cup H_2$ is a subgroup, therefore $ab = c$ (say) is also an element of $H_1 \cup H_2$.

But $ab = c \in H_1 \cup H_2 \Rightarrow ab = c \in H_1$ or H_2. 1

Suppose $ab = c \in H_1$.

Then $b = a^{-1}c \in H_1$ [$\because$ H_1 is a subgroup, therefore $a \in H_1 \Rightarrow a^{-1} \in H_1$] 1

But from (2), we have $b \notin H_1$. Thus we get a contradiction.

Again suppose $ab = c \in H_2$.

Then $a = cb^{-1} \in H_2$ [$\because$ H_2 is a subgroup, therefore $b \in H_2 \Rightarrow b^{-1} \in H_2$] 1

But from (1), we have $a \notin H_2$. Thus here also we get a contradiction.

Hence either $H_1 \subseteq H_2$ or $H_2 \subseteq H_1$. 1

ii. Let G be a group, $a, b \in G$ such that $o(a) = m$, $o(b) = n$ and $ab = ba$. If $(m, n) = 1$ then $o(ab) = mn$.

Ans

$$\left.\begin{array}{l} \text{Since } o(a) = n \Rightarrow a^n = e \\ o(b) = m \Rightarrow b^m = e \end{array}\right\} \quad (*)$$

Consider $(ab)^{mn} = (ab)\cdot(ab)\ldots(ab)$ 1

$= (a \cdot a \ldots a)\cdot(b \cdot b \ldots b)$ as $ab = ba$

$= a^{mn} \cdot b^{mn}$

$= (a^n)^m \cdot (b^m)^n$

$= e^m \cdot e^n$ by (*)

$= e$ 2

Let $o(ab) = r$

$r \mid mn$ ... (1)

$\Rightarrow (ab)^r = e$

$a^r \cdot b^r = e$ (as $ab = ba$)

$a^r = b^{-r}$ 2

$\therefore (a^r)^m = (b^{-r})^m$

$\Rightarrow (a^r)^m = (b^m)^{-r} = e$ (as $o(b) = m$) 1

$\Rightarrow (a)^{rm} = e$ and $o(a) = n$

$\Rightarrow n \mid rm \Rightarrow n \mid r$ as $(n, m) = 1$.

Similarly we can show $m \mid r$.

As $(n, m) = 1 \Rightarrow nm \mid r$ ... (2) 1

$\therefore$ By (1), (2) $r = mn = o(ab)$ 1

Q.2 Attempt any **TWO** questions from the following: (12)

$b)$ i. Show that $(\mathbb{Q}^*, o)$ is a group, where $a\ o\ b = \frac{ab}{6}$, for $a, b \in \mathbb{Q}^*$.

Ans Closure 2

Associative trivial

identity $= e = 6$ 2

Inverse of $a \in \mathbb{Q}^*$ is $b = \frac{36}{a}$ 2

ii. Let G be an Abelian group and n be a fixed positive integer.

Let $G^n = \{g^n | g \in G\}$. Prove that G^n is a subgroup of G.

Ans Let $g_1^n, g_2^n \in G^n$

$g_1^n(g_2^n)^{-1} = g_1^n(g_2^{-1})^n = (g_1 g_2^{-1})^n$ ($\because$ G is abelian) 2

Hence, $g_1^n(g_2^n)^{-1} \in G^n$ ($\because g_1 g_2^{-1} \in G$) 2

By 1- step test G^n is a subgroup of G. 2

iii. Let $\alpha = (1\ 2\ 6)(6\ 3\ 4)(5\ 6\ 2)$ and $\beta = (1\ 5\ 4)(3\ 1\ 2\ 4)$ in S_6

p) Write α and β as a product of disjoint cycles

q) Is $O(\alpha\beta) = O(\alpha)O(\beta)$? Justify.

Ans p) $\alpha = (1\ 2\ 5\ 3\ 4\ )$, $\beta = (1\ 2\ )(3\ 5\ 4\ )$ 2

q) $O(\alpha) = 5$, $O(\beta) = 2 \times 3 = 6 \implies O(\alpha)\ O(\beta) = 30$
$\alpha\beta = (1\ 5\ ) \implies O(\alpha\beta) = 2$
$\therefore O(\alpha\beta) \neq O(\alpha)O(\beta)$
Note that α and β are not disjoint . 4

iv. If $a^2 = e$, $\forall\, a \in G$ then show that G is abelian group.

Ans $(ab)(ba) = ab^2a = aea = a^2 = e$ 2

But $(ab)(ab) = e$ 2

By uniqueness of inverse $ab = ba$ 2

Q3. Attempt any **ONE** question from the following: (08)

$a)$ i. Let G be a finite cyclic group of order 'n' then prove that G has unique subgroup of order $'d'$ for each divisor d of n.

Ans G is a finite cyclic group of order 'n' generated by $'a'$
$G = < a >, O(a) = n$
Let $d | n \therefore n = dd_1$

Consider $H = < a^{\frac{n}{d}} > = < a^{d_1} >$

$O(a^{d_1}) = \frac{n}{(n, d_1)} = \frac{n}{d_1} = d$

$\therefore H = < a^{\frac{n}{d}} >$ is a subgroup of order d 4

Uniqueness:

Let H' be any other subgroup of order d

We know that H' is generated by a^m where m is the smallest positive integer such that $a^m \in H'$

$H' = < a^m >$

$\exists!\, q, r\ s.t\ n = mq + r, where\ \ r = 0\ or\ r < m$

If $r < m$ then $a^r = (a^m)^{-q} \in H'$

Which is a contradiction because m is the smallest positive integer such that $a^m \in H'$

$n = mq$

$O(H') = d$

$O(a^m) = d$

$\frac{n}{(n, m)} = d$

$\frac{n}{m} = d$ 4

$\therefore H' = < a^m > = < a^{\frac{n}{d}} > = H$

ii. List all generators and all subgroups of the cyclic group $G = \langle a \rangle$ of order 16.

Ans $G = < a >, O(G) = O(a) = 16$

Generators

$$a^1, a^3, a^5, a^7, a^{11}, a^{13}, a^{15}$$ 2

Subgroups

$1|16\ \ \exists!$ subgroup of order 1 namely

$H_1 = < a^{\frac{16}{1}} > = \{e\}$

$2|16\ \ \exists!$ subgroup of order 2 namely

$H_2 = < a^{\frac{16}{2}} > = < a^8 > = \{a^8, e\}$

$4|16\ \ \exists!$ subgroup of order 4 namely

$H_3 = < a^{\frac{16}{4}} > = < a^4 > = \{a^4, a^8, a^{12}, e\}$

$8|16\ \ \exists!$ subgroup of order 8 namely

$H_4 = < a^{\frac{16}{8}} > = < a^2 > = \{a^2, a^4, a^6, a^8, a^{10}, a^{12}, a^{14}, e\}$

$16|16\ \ \exists!$ subgroup of order 16 namely

$H_5 = < a^{\frac{16}{16}} > = < a^1 > = G$ 6

Q3. Attempt any **TWO** questions from the following: (12)

$b)$ i. If G is infinite cyclic group generated by a then show that G has exactly two generators $'a'$ and $'a^{-1}'$.

Ans G is infinite cyclic group generated by a

Let b be a generator of G

$\therefore < a > = < b >$

$a \in < a > \subseteq < b >$ $\therefore a = b^n$, for some $n \in \mathbb{Z}$

$b \in < b > \subseteq < a >$ $\therefore b = a^m$, for some $m \in \mathbb{Z}$

$a = b^n = (a^m)^n = a^{mn}$

$\therefore mn = 1$ ($\because a$ is of infinite order)

$m, n = 1 \; or - 1$

$b = a^1 \;\; or \;\; a^{-1}$

$\therefore G$ has exactly two generators $'a'$ and $'a^{-1}'$ 6

ii. Let $U(n) = \{\bar{x} \mid x \in \mathbb{N}, (x, n) = 1, 1 \le x \le n\}$ under multiplication modulo n. Determine which of the following groups are cyclic. Justify your answer.

(p) $U(5)$ (q) $U(8)$

Ans $U(5) = \{1,2,3,4\}$

$U(8) = \{1,3,5,7\}$ 2

As $O(2) = 4$ $\therefore U(5) = < 2 >$ 2

As $O(3) = O(5) = O(7) = 2$ $\therefore U(8)$ is not cyclic 2

iii. Prove that every subgroup of a cyclic group is cyclic.

Ans Let H be a subgroup of a cyclic group $G = < a >$

Claim: H is generated by a^m where m is the smallest positive integer such that $a^m \in H$ 2

T.P.T $H = < a^m >$

$H \supseteq < a^m >$ ….(1) 1

T.P.T $H \subseteq < a^m >$

Let $b = a^k \in H \; for \; some \; k$

$\exists! \, q, r \; s.t \; k = mq + r, where \;\; r = 0 \; or \; r < m$

If $r < m$ then $a^r = a^{k-mq} = a^k (a^m)^{-q} \in H$ which is a contradiction because m is the smallest positive integer such that $a^m \in H$

$r = 0$

$k = mq$

$b = a^k = a^{mq} \in < a^m >$

$H \subseteq < a^m >$….(2)

$H = < a^m >$ ⋯from (1) and (2) 3

∴Every subgroup of a cyclic group is cyclic

iv. Prove or disprove: $G = \{(1,1), (1,-1), (-1,1), (-1,-1)\}$ is a group under the operation $(a, b)(c, d) = (ac, bd)$ but not a cyclic group.

Ans

	(1,1)	(1,−1)	(−1,1)	(−1,−1)
(1,1)	(1,1)	(1,−1)	(−1,1)	(−1,−1)
(1,−1)	(1,−1)	(1,1)	(−1,−1)	(−1,1)
(−1,1)	(−1,1)	(−1,−1)	(1,1)	(1,−1)
(−1,−1)	(−1,−1)	(−1,1)	(1,−1)	(1,1)

4

G is closed under the given operation
Identity element = e = (1,1)
Inverse of each element is itself
$\therefore G$ is a group under the given operation 1

As order of each element is 2
$\therefore G$ is not cyclic 1

Q4. Attempt any **ONE** question from the following: (08)

a) i. State and Prove Lagrange's theorem for finite group.

Ans Statement : Let G be a finite group and H is subgroup of G then $o(H)|o(G)$. 1M

Proof : Since G is finite group it has finite number of left (or right) cosets. 2M

Claim1 : $f: H \to aH$ is bijective using $f(h) = ah\,, h \in H$

Claim1 gives $o(aH) = o(H), \forall a \in G$ --------------(1)

Claim2 : Any two cosets are either identical or disjoint. 2M

Suppose $a_1H, a_2H, \cdots\cdots\cdots, a_kH$ are all distinct and disjoint cosets of H in G. ------(2)

Claim3 : $G = \bigcup_{i=1}^{k} a_iH$ 2M

Using (2) and claim3 we get $o(G) = o(a_1H) + o(a_2H) + \cdots\cdots\cdots + o(a_kH)$ 1M

Using (1) $o(G) = o(H) + o(H) + \cdots\cdots\cdots + o(H), k$ −times

$o(G) = k\, o(H) \implies o(H)|o(G)$.

ii. Let $f: G \longrightarrow G'$ is onto group homomorphism. then show that

(p) $f(e) = e'$, where e and e' are identities of G and G' respectively.

(q) $f(a^{-1}) = [f(a)]^{-1}, \forall\, a \in G$

(r) $f(a^m) = [f(a)]^m,\ \forall\, a \in G\,, m \in \mathbb{N}$

Ans (p) Since $e \cdot e = e \implies f(e \cdot e) = f(e) \implies f(e) \cdot f(e) = f(e) \cdot e'$

using LCL we get $f(e) = e'$ 2M

(q) $\because\ a\,a^{-1} = e \implies f(a \cdot a^{-1}) = f(e)$

$\implies f(a) \cdot f(a^{-1}) = e' \implies f(a^{-1}) = [f(a)]^{-1}$ 2M

(r) using induction, For $m = 1$, $f(a^1) = f(a)$ and $[f(a)]^1 = f(a)$

Suppose for $m = k$, $f(a^k) = [f(a)]^k$

Consider $f(a^{k+1}) = f(a^k)\, f(a) = [f(a)]^k\, f(a) = [f(a)]^{k+1}$

Hence $f(a^m) = [f(a)]^m,\ \forall\, a \in G, m \in \mathbb{N}$ 4M

Q4. Attempt any **TWO** questions from the following: (12)

$b)$ i. Let G be a group of prime order p. If H and K are subgroups of G then show that either $H \cap K = \{e\}$ or $H = K$.

Ans Since H and K be two subgroups of G.

By Lagrange's theorem , $o(H) | o(G)$ and $o(K) | o(G) \implies o(H) | p$ and $o(K) | p$

As p is prime. $o(H) = o(K) = 1$ *or* p

If $o(H) = o(K) = 1 \implies H = K = \{e\}$ ---------- (1)

If $o(H) = o(K) = p = o(G)$, also H *and* $K \subseteq G$ gives $H = K = G$ -- (2)

(1) and (2) gives either $H \cap K = \{e\}$ or $H = K$. 6M

ii. Let G be a group of order pq where p and q are distinct prime integers. Show that every subgroup $H \neq G$ is a cyclic subgroup of G.

Ans Since H is subgroups of G

By Lagrange's theorem , $o(H) | o(G) \implies o(H) | pq$

As p and q are distinct primes and $H \neq G \implies o(H) = 1$ or p or q 3M

If $o(H) = 1 \implies H = \{e\} \implies H$ is cyclic. 1M

If $o(H) = p$ and p is prime $\implies H$ is cyclic. 1M

If $o(H) = q$ and q is prime $\implies H$ is cyclic. 1M

iii. Find the number of group homomorphism from $\mathbb{Z}_{20}$ to $\mathbb{Z}_8$.

Ans Let $f: \mathbb{Z}_{20} \rightarrow \mathbb{Z}_8$ is a group homomorphism.

Let $\bar{1} \in \mathbb{Z}_{20}$ then $f(\bar{1})$ defines all homomorphism. Also $o(\bar{1}) = 20$

$\bar{1} \in \mathbb{Z}_{20}$ and $f(\bar{1}) \in \mathbb{Z}_8 \implies o\big(f(\bar{1})\big) \mid o(\mathbb{Z}_8) \implies o\big(f(\bar{1})\big) \mid 8$ ------- (1)

As f is homomorphism $\implies o\big(f(\bar{1})\big) \mid o(\bar{1}) \implies o\big(f(\bar{1})\big) \mid 20$ ------- (2)

(1) and (2) $\implies o\big(f(\bar{1})\big) \mid \gcd(20, 8) \implies o\big(f(\bar{1})\big) \mid 4$

$\Rightarrow o(f(\bar{1})) = 1, 2 \text{ or } 4$ 6M

Since $f(\bar{1}) \in \mathbb{Z}_8 \Rightarrow$ there are 4 elements $\{\bar{0}\}$, $\{\bar{4}\}$, $\{\bar{2}, \bar{6}\}$ of order $1, 2, 4$ respectively. Hence there are **four** homomorphism.

iv. Define automorphism of groups. Let $a \in G$, show that $f_a: G \longrightarrow G$ defined by $f_a(x) = axa^{-1}, \forall\, x \in G$ is an automorphism.

Ans An automorphism is an isomorphism from a group G to itself. 1M

Now $f_a(x)\, f_a(y) = (axa^{-1})\,(aya^{-1}) = axa^{-1}aya^{-1}$

$$= a(xy)a^{-1} = f_a(xy)$$

Now $f_a(x) = f_a(y) \Rightarrow axa^{-1} = aya^{-1} \Rightarrow x = y$ using LCL & RCL

Let $y \in G$ and $a \in G$ then $a^{-1}ya \in G$ such that

$f_a(a^{-1}ya) = aa^{-1}yaa^{-1} = y$ 5M

Therefore f_a is an automorphism.

Q5. Attempt any **FOUR** questions from the following: (20)

$a)$ State and prove necessary and sufficient condition for a non empty subset H of a group G to be a subgroup of G.

Ans A non empty subset H of a group G is a subgroup of G if and only if $ab^{-1} \in H$ for $a, b \in H$ 2

Identity
$a \in H \therefore aa^{-1} \in H \therefore e \in H$ 1

Inverse
$e, a \in H \therefore ea^{-1} = a^{-1} \in H$ 1

Closure
$for\ a, b \in H \Rightarrow a, b^{-1} \in H \Rightarrow a(b^{-1})^{-1} \in H \Rightarrow ab \in H$

1

$b)$ Let $G = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$. Construct composition table for G under multiplication of 2×2 matrices.

Ans

	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$
$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

5

$c)$ Prove that every cyclic group is abelian.

Ans Let G be a cyclic group generated by $'a'$

Let $x, y \in G$

$\therefore x = a^i$, for some $i \in \mathbb{Z}$ and $y = a^j$,for some $j \in \mathbb{Z}$

$xy = a^i a^j = a^{i+j} = a^{j+i} = a^j a^i = yx$

$\therefore G$ is abelian 5

$d)$ Let G be a cyclic group of order 42. Find the number of elements of order 6 and the number of elements of order 7 in G. Clearly state the result used.

Ans Result :

If G is a cyclic group of order n generated by a then for every divisor d of n there are $\varphi(d)$ elements of order d 2

$6|42$ $\therefore$number of elements of order 6 $=\varphi(6) = 2$

$7|42$ $\therefore$number of elements of order 7 $=\varphi(7) = 6$

3

$e)$ Prove that every group of order 49 contains a subgroup of order 7.

Ans Let G be a group of order 49.

If G is cyclic then $\exists\, a \in G$ such that $o(a) = 49$

Now $o(a^7) = \frac{o(a)}{(o(a)\,,\ 7)} = \frac{49}{(49,7)} = 7$ then $H = \langle a^7 \rangle$ is subgroup of G of order 7

If G is not cyclic then G doesn't have any element of order 49.

Also $o(a)|\, o(G)$, $\forall\, a \in G$ then $\exists\, b \in G$, $b \neq e$ such that $o(b) = 7$

Hence $K = \langle b \rangle$ is a subgroup of G of order 7.

6M

$f)$ Let G be a group. Show that $f: G \longrightarrow G$ defined by $f(x) = x^2, \forall x \in G$ is an homomorphism if and only if

Ans Since f is homomorphism then $f(xy) = f(x)f(y) \implies (xy)^2 = x^2\, y^2$

$\implies (xy)(xy) = xx\, yy \implies x(yx)y = x(xy)y \implies yx = xy \implies G$ is abelian. 3M

Conversely, as G is abelian $f(xy) = (\boldsymbol{xy})^{\mathbf{2}} = \boldsymbol{x}^{\mathbf{2}}\, \boldsymbol{y}^{\mathbf{2}} = f(x)f(y)$

Therefore f is homomorphism. 2M