Q.P. Code: 33413

(2½ Hours)

[Total Marks: 75]

N. B.: (1) **All** questions are **compulsory**.
(2) Make **suitable assumptions** wherever necessary and **state the assumptions** made.
(3) Answers to the **same question** must be **written together**.
(4) Numbers to the **right** indicate **marks**.
(5) Draw **neat labelled diagrams** wherever **necessary**.
(6) Use of **Non-programmable** calculators is **allowed**.

1. Attempt *any three* of the following: 15

a. State the characteristics of typical mathematical models of physical world. Explain with example.

b. Discuss the conservation laws and engineering with respect to mathematical models.

c. Suppose that you have the task of measuring the lengths of a bridge and a rivet and come up with 9999 and 9 cm, respectively. If the true values are 10,000 and 10 cm, respectively, compute (i) the true error and (ii) the true percent relative error for each case.

d. Use zero- through third-order Taylor series expansions to predict f (3) for

$$f(x) = 25x^3 - 6x^2 + 7x - 88$$

using a base point at $x = 1$.

Ans 554

e. Determine the absolute and relative errors when approximating p by p^* when

i. $p = 0.3000 \times 10^1$ and $p^* = 0.3100 \times 10^1$
ii. $p = 0.3000 \times 10^{-3}$ and $p^* = 0.3100 \times 10^{-3}$
iii. $p = 0.3000 \times 10^4$ and $p^* = 0.3100 \times 10^4$

Ans (i) For $p = 0.3000 \times 10^1$ and $p^* = 0.3100 \times 10^1$ the absolute error is 0.1, and the relative error is 0.3333×10^{-1}.

(ii) For $p = 0.3000 \times 10^{-3}$ and $p^* = 0.3100 \times 10^{-3}$ the absolute error is 0.1×10^{-4}, and the relative error is 0.3333×10^{-1}.

(ii) For $p = 0.3000\times10^4$ and $p^* = 0.3100\times10^4$, the absolute error is 0.1×10^3, and the relative error is again 0.3333×10^{-1}.

f. Let p = 0.54617 and q = 0.54601. Use four-digit arithmetic to approximate $p - q$ and determine the absolute and relative errors using (i) rounding and (ii) chopping.

Ans The exact value of $r = p - q$ is $r = 0.00016$.

Suppose the subtraction is performed using four-digit rounding arithmetic. Rounding p and q to four digits gives $p^* = 0.5462$ and $q^* = 0.5460$, respectively, and $r^* = p^* - q^* = 0.0002$ is the four-digit approximation to r. Since

$$\frac{|r - r^*|}{|r|} = \frac{|0.00016 - 0.0002|}{|0.00016|} = 0.25,$$

the result has only one significant digit, whereas p^* and q^* were accurate to four and five significant digits, respectively.

If chopping is used to obtain the four digits, the four-digit approximations to p, q, and r are $p^* = 0.5461$, $q^* = 0.5460$, and $r^* = p^* - q^* = 0.0001$. This gives

$$\frac{|r - r^*|}{|r|} = \frac{|0.00016 - 0.0001|}{|0.00016|} = 0.375,$$

which also results in only one significant digit of accuracy. ■

2. Attempt *any three* of the following: **15**

a. Use the Bisection method to find solutions accurate to within 10^{-2} for $x^3 - 7x^2 + 14x - 6 = 0$ in the interval [3.2,4].

Ans 3.419

b. The fourth-degree polynomial $f(x) = 230x^4 + 18x^3 + 9x^2 - 221x - 9$ in [0, 1]correct upto 4 decimal places using Regula-Falsi method.

Ans 0.96239

c. Find the root of $4x^2 - e^x - e^{-x} = 0$ using Newton Raphson correct upto 4 decimal places using initial value as 1.

Ans 0.824498585

d. Given the cube of integers in the following table. Find the values of $(5.5)^3$ and 15^3 using Newton's interpolation formula.

Ans 166.375 and 3375

e. Find $f(0.9)$ if $f(0.6) = -0.17694460$, $f(0.7) = 0.01375227$, $f(0.8) = 0.22363362$, $f(1.0) = 0.65809197$ using Lagrange's Interpolation formula.

Ans 0.44198500

f. Using appropriate interpolation formula find $f(4.25)$ from the table:

X	4.0	4.1	4.2	4.3	4.4	4.5
f(x)	27.21	30.18	33.35	36.06	40.73	54.01

Ans 35.059

3. Attempt *any three* of the following: **15**

a. Solve the following system by using the Gauss-Jordan elimination method.

$$a + b + 2c = 1$$
$$2a - b + d = -2$$
$$a - b - c - 2d = 4$$
$$2a - b + 2c - d = 0$$

Ans 1,2,-1,-2

b. Solve the following system by using the Gauss-Seidel iterative method.

$$10a - b + 2c = 6$$
$$-a + 11b - c + 3d = 25$$

$$2a - b + 10c - d = -11$$
$$3b - c + 8d = 15$$

Ans 1,2,-1,1

c. Find $\left(\frac{dy}{dx}\right)_{x=5.2}$, if

x	4.9	5.0	5.1	5.2	5.3	5.4	5.5
y	134.290	148.413	164.022	181.272	200.337	221.406	244.692

Ans 18.148

d. Evaluate $\int_0^{0.3} \sqrt{1 - 8x^2}dx$ using Simpson's 3/8th rule.

Ans 0.2916

e. Apply Taylor's method of order two with *N* = 10 to the initial-value problem

$$y' = y - t^2 + 1, 0 \le t \le 2 \le, y(0) = 0.5$$

Ans

t_i	Taylor Order 2 w_i
0.0	0.500000
0.2	0.830000
0.4	1.215800
0.6	1.652076
0.8	2.132333
1.0	2.648646
1.2	3.191348
1.4	3.748645
1.6	4.306146
1.8	4.846299
2.0	5.347684

f. Using modified Euler's method find the solution of

$$y' = cos2t + sin3t), \quad 0 \le t \le 1; y(0) = 1 \ \ with\ h = 0.25$$

Ans

t	Modified Euler	$y(t)$
0.25	1.3199027	1.3291498
0.50	1.7070300	1.7304898
0.75	2.0053560	2.0414720
1.00	2.0770789	2.1179795

4. Attempt *any three* of the following: **15**

a. Fit an exponential model to:

x	0.4	0.8	1.2	1.6	2.0	2.3
y	800	975	1500	1950	2900	3600

Ans

1. *An exponential model has equation* $y = ae^{bx}$. *First we want to linearize the equation* $y = ae^{bx}$ *with a natural log* ($\ln$) *transform. This gives* $\ln(y) = \ln(a) + bx$.

2. *Now we apply the* $\ln$ *transform to get a table of linearized data:*

x	y	$\ln(y)$
0.4	800	6.685
0.8	975	6.882
1.2	1500	7.313
1.6	1950	7.576
2	2900	7.972
2.3	3600	8.187

3. *We calculate the regression line* $\ln(y) = a_0 + a_1 x$ *for* x *and* $\ln(y)$ *using the least squares method. We use the equations:*

$$a_1 = \frac{n\sum x_i y_i - \sum x_i \sum y_i}{n\sum x_i^2 - (\sum x_i)^2}$$

$$a_0 = \bar{y} - a_1 \bar{x}$$

We need to calculate the sums in these equations and also the means $\bar{x}$ *and* $\bar{y}$ *(remember that we will be using* $\ln(y)$ *in place of* y*). The easiest way to do this is to expand our table:*

x	y	$\ln(y)$	x^2	$x\ln(y)$
0.4	800	6.685	0.16	2.674
0.8	975	6.882	0.64	5.506
1.2	1500	7.313	1.44	8.776
1.6	1950	7.576	2.56	12.122
2	2900	7.972	4	15.944
2.3	3600	8.187	5.29	18.83
8.3	NA	44.615	14.09	63.852

4. *We use the values from the table to get* $\bar{x} = \frac{8.3}{6} = 1.383$ *and* $\overline{\ln(y)} = \frac{44.615}{6} = 7.436$. *We use these values and the sums from the table to get*

$$a_1 = \frac{6(63.852) - (8.3)(44.615)}{6(14.09) - (8.3)^2} = 0.818$$

$$a_0 = 7.436 - 0.818(1.383) = 6.3$$

5. *Comparing the linearized exponential equation with the regression line for the transformed data gives*

$$\ln(a) + bx = 6.3 + 0.818x$$

Equating coefficients gives $\ln(a) = 6.3$, *and so* $a = e^{6.3}$, *and* $b = 0.818$. *So the exponential model is* $y = e^{6.3}e^{0.818x}$.

b. Find the least square polynomial approximation of degree two to the data

x	0	1	2	3	4
y	–4	–1	4	11	20

Ans **Sol.** Let the equation of the polynomial be $y = a + bx + cx^2$ ...(1)

x	y	xy	x^2	x^2y	x^3	x^4
0	−4	0	0	0	0	0
1	−1	−1	1	−1	1	1
2	4	8	4	16	8	16
3	11	33	9	99	27	81
4	20	80	16	320	64	256
$\sum x = 10$	$\sum y = 30$	$\sum xy = 120$	$\sum x^2 = 30$	$\sum x^2y = 434$	$\sum x^3 = 100$	$\sum x^4 = 354$

The normal equations are,

$$\sum y = na + b\sum x + c\sum x^2 \quad ...(2)$$

$$\sum xy = a\sum x + b\sum x^2 + c\sum x^3 \quad ...(3)$$

$$\sum x^2y = a\sum x^2 + b\sum x^3 + c\sum x^4 \quad ...(4)$$

Here $n = 5, \sum x = 10, \sum y = 30, \sum xy = 120, \sum x^2 = 30, \sum x^2y = 434, \sum x^3 = 100,$ $\sum x^4 = 354.$

Putting all these values in (2), (3) and (4), we get

$$30 = 5a + 10b + 30c \quad ...(5)$$

$$120 = 10a + 30b + 100c \quad ...(6)$$

$$434 = 30a + 100b + 354c \quad ...(7)$$

On solving these equations, we get $a = -4$, $b = 2$, $c = 1$. Therefore, required polynomial is $y = -4 + 2x + x^2$.

c. Find the best-fit values of a and b so that $y = a + bx$ fits the data given in the table.

x	0	1	2	3	4
y	1	1.8	3.3	4.5	6.3

Ans $y = 0.72 + 1.33x$

d. A painter has exactly 32 units of yellow dye and 54 units of green dye. He plans to mix as many gallons as possible of color A and color B. Each gallon of color A requires 4 units of yellow dye and 1 unit of green dye. Each gallon of color B requires 1 unit of yellow dye and 6 units of green dye. Find the maximum number of gallons he can mix graphically.

Ans the objective function is to determine the maximum number of gallons he can mix.

the colors involved are color A and color B.

let x = the number of gallons of color A.
let y = the number of gallons of color B.

if we let g = the maximum gallons the painter can make, then the objective function becomes:

g = x + y

make a table for color A and color B to determine the amount of each dye required.
your table will look like this:

each gallon of color A or B will require:

	units of yellow dye	units of green dye
color A	4	1
color B	1	6

total units of yellow dye available are 32
total units of green dye available are 54

your constraint equations are:
x >= 0
y >= 0
4x + y <= 32
x + 6y <= 54

x and y have to each be greater than or equal to 0 because the number of gallons can't be negative.

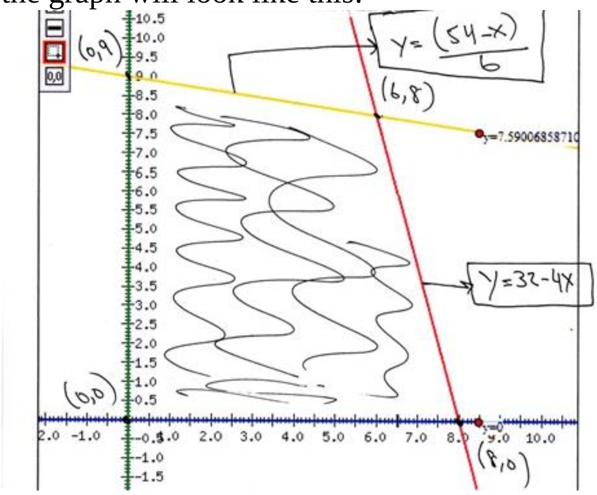
in order to graph these equations, you solve for y in those equations that have y in them.

the equations for graphing are:

x >= 0
y >= 0
y <= 32 - 4x
y <= (54 - x)/6

x = 0 is a vertical line that is the same line as the y-axis.
y = 0 is a horizontal line that is the same line as the x-axis.

the graph will look like this:



the region of feasibility is the shaded area of the graph.
all points within the feasibility region meet the constraint of the problem.

the intersection points of the region of feasibility are:

(0,0)
(0,9)
(6,8)
(8,0)

the maximum or minimum value of the objective function will be at these points of intersection.

solve the objective function at each of these intersection points to determine which point contains the maximum number of gallons.

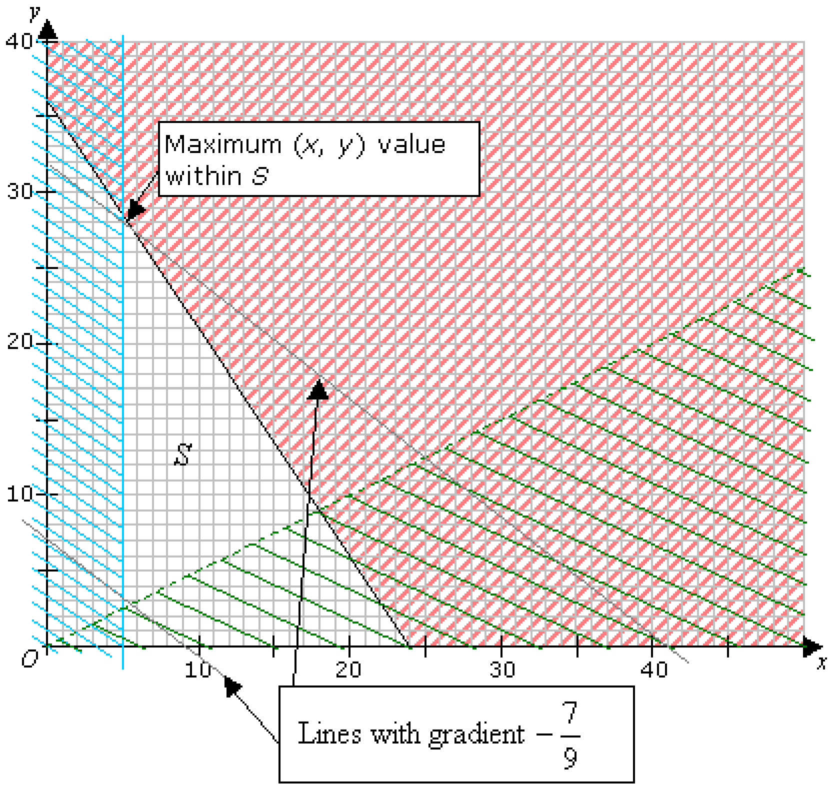
the objective function is:
g = x + y
the table with the value of g at each of these intersection points is shown below:

intersection point (x,y)	gallons of paint
(0,0)	0
(0,9)	9
(6,8)	14 *****
(8,0)	8

the maximum gallons of paint for color A and B, given the constraints, is equal to 14.
this is comprised of 6 gallons of color A and 8 gallons of color B.
6 gallons of color A uses 24 gallons of yellow dye and 8 gallons of color B uses 8 gallons of yellow dye for a total of 32 gallons of yellow dye which is the maximum amount of yellow dye that can be used.
6 gallons of color A user 6 gallons of green dye and 8 gallons of color B uses 48 gallons of green dye for a total of 54 gallons of green dye which is the maximum amount of green dye that can be used.

e. Rita wants to buy *x* oranges and *y* peaches from the store. She must buy at least 5 oranges and the number of oranges must be less than twice the number of peaches. An orange weighs150 grams and a peach weighs 100 grams. Joanne can carry not more than 3.6 kg of fruits home.
   i) Write 3 inequalities to represent the information given above.
   ii) Plot the inequalities on the Cartesian grid and show the region that satisfies all the inequalities. Label the region *S*.
   iii) Oranges cost ₹ 0.70 each and peaches cost ₹ 0.90 each. Find the maximum that Rita can spend buying the fruits.

Ans At least 5 oranges: $x \geq 5$
oranges less than twice of peaches: $x < 2y$
not more than 3.6 kg: $150x + 100y \leq 3600 \Rightarrow 3x + 2y \leq 72$



The maximum value is found at (5,28) i.e. 5 oranges and 28 peaches. Therefore, the maximum that Rita can spend on the fruits is: $0.70 \times 5 + 0.90 \times 28 = 28.70$.

f. Consider a calculator company which produces a scientific calculator and a graphing calculator. Long-term projections indicate an expected demand of at least 1000 scientific and 800 graphing calculators each month. Because of limitations on production capacity, no more than 2000 scientific and 1700 graphing calculators can be made monthly. To satisfy a supplying contract, a total of atleast 2000 calculators must be supplied each month. If each scientific calculator sold results in Rs.120 profit and each graphing calculator sold produces a Rs.150 profit, how many of each type of calculators should be made monthly to maximize the net profit?

Ans Suppose, x is the number of scientific calculators produced and y is the number of graphing calculators produced. Since the company can not produce negative calculators; it will produce either no or positive numbers of calculators only, we must have $x \geq 0$, $y \geq 0$. But, keeping in mind the demand of each type of calculator we have

$x \geq 1000$

$y \geq 800$.

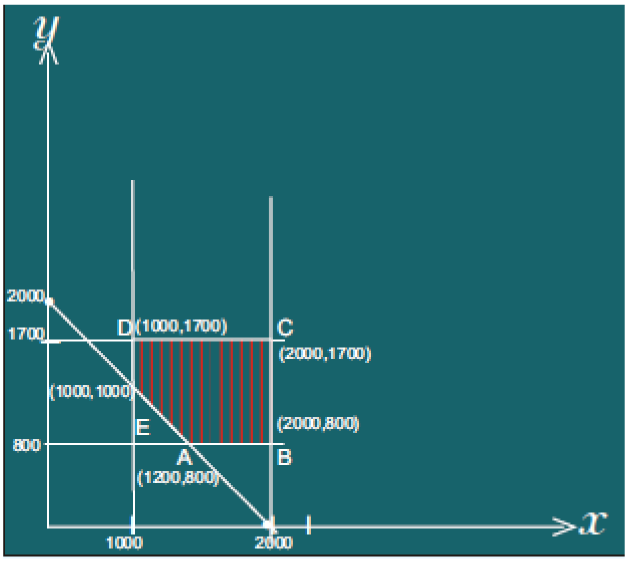
$x \leq 2000$

$y \leq 1700$.

$x + y \geq 2000$.

The net profit from sale of x scientific and y graphing calculators is $120x + 150y$

the ultimate model of the problem becomes max $120x + 150y$ subject to $1000 \leq x \leq 2000$, $800 \leq y \leq 1700$, $x + y \geq 2000$.



the maximum value of 120x + 150y is 510,000 at (2000,1700).

5. **Attempt *<u>any three</u>* of the following:** **15**

a. The mileage C in thousands of miles which car owners get with a certain kind of tyre is a random variable having probability density function

$$f(x) = \frac{1}{20}e^{-\frac{x}{20}} \qquad for\ x > 0$$
$$= 0, \qquad for\ x \leq 0$$

Find the probabilities that one of these tyres will-last

i. At most 10000 miles
ii. Anywhere from 16000 to 24000 miles
iii. At least 30000 miles

Ans 0.3935, 0.1481, 0.2231

b. A petrol pump is supplied with petrol once a day. If its daily volume *X* of sales in thousands of litres is distributed by

$$f(x) = 5(1-x)^4, 0 \leq x \leq 1$$

what must be the capacity of its tank in order that the probability that its supply will be exhausted in a given day shall be 0.01?

Ans 601.9 litres

c. A continuous random variable X has a p.d.f.

$$f(x) = 3x^2, \quad 0 \leq x \leq 1$$

Find *a* and *b* such that

i. $P(X \leq a) = P(X > a)$ and
ii. $P(X > b) = 0.05$

And $(1/2)^{1/3}$, $(19/20)^{1/3}$

d. What is the probability of getting a total of 9 (i) twice and (ii) at least twice in 6 tosses of a pair of dice?

Ans 61440/531441, 72689/531441

e. In a precision bombing attack there is a 50% chance that any one bomb will strike the target. Two direct hits are required to destroy the target completely. How many bombs must be dropped to give a 99% chance or better of completely destroying the target?

Ans 11

f. A car hire firm has two cars which it fires out day by day. The number of demands for a car on each day is distributed as Poisson variate with mean 1.5. Calculate the proportion of days on which (i) neither car is used, and (ii) some demand is refused.

Ans 0.2231, 0.19126