M.Sc. (Mathematics) Part – II

Algebra – II

Paper – I April: - 2016

QP Code : 25037

Scheme A (External)]	(3 Hours)	**[Total Marks:100**
Scheme B (Internal)]	(2 Hours)	**[Total Marks: 40**

Instructions:

- **Mention** on the top of the **answer book** the **scheme** under which you are appearing
- **Scheme A** students should **attempt any five** questions
- **Scheme B** students should **attempt any three** questions
- All **questions** carry **equal marks**

1. (a) Let G be a finite group and let p be a prime dividing the order of G. Prove that any two Sylow-p-subgroups are conjugates. Next, deduce that the number of all Sylow-p-subgroups of G is a divisor of $n = o(G)$.

   (b) Show that a group of order 35 is cyclic.

2. (a) Show that a nilpotent group is solvable. Is the converse true? Justify your answer.

   (b) Show that a group of order $p.q$ where p and q are primes, is solvable.

3. (a) Let $f(x)$ be a monic polynomial in $F[x]$.Prove that $f(x)$ has a multiple root in a splitting field of $f(x)$ if and only if its derivative $f'(x) = 0$. Hence show that if char $F = 0$, then $f(x)$ has no multiple roots and if char $F = p > 0$ then $f(x)$ has multiple roots if and only $f(x)$ is in $F[x^p]$

   (b) Define cyclotomic polynomial $\phi_n(x)$ and show that it is an irreducible polynomial in $\mathbb{Q}[X]$.

4. (a) For any prime number p and for $n \in \mathbb{N}$, show that there exists a finite field of order p^n and is unique upto isomorphism.

   (b) Let F be a finite field. Show that $F^* = F - \{0\}$ is a cyclic group under multiplication.

5. (a) Show that a regular n-gon is constructible if and only if $\phi(n)$ is a power of 2 where $\phi(n)$ is number of positive integers less than n and relatively prime to n.

   (b) Find the Galois group of $x^3 - 2$ over $\mathbb{Q}$.

6. (a) Let E be the splitting field of a polynomial $f(x)$ over a field F. Show that $|Aut(E/F)| \leq [E : F]$ where $Aut(E/F)$ denotes the group of automorphisms of field E which fixes F element wise.

   (b) Consider $f(x) = x^5 - 1$ in $\mathbb{Q}[x]$. Find the splitting field of $f(x)$ over $\mathbb{Q}$ and its Galois group.

7. (a) Define Noetherian ring. If R is a Noetherian ring show that the polynomial ring $R[X]$ is also Noetherian.

   (b) Show that every abelian group is a $\mathbb{Z}$-module.

8. (a) Show that every finitely generated module over a PID is the direct sum of a free module and a torsion module.

   (b) If N is a submodule of M with N and M/N finitely generated modules, show that M is also finitely generated.

M.Sc. (Mathematics) Part – II
Analysis - II
Paper - II
April: - 2016

QP Code : 25047

Scheme A (External)]	(3 Hours)	[Total Marks:100
Scheme B (Internal)]	(2 Hours)	[Total Marks: 40

Instructions:

- **Mention** on the top of the **answer book** the **scheme** under which you are appearing
- **Scheme A** students should **attempt any five** questions
- **Scheme B** students should **attempt any three** questions
- All **questions** carry **equal marks**

1. (a) Define outer (Exterior) measure of a set. Prove or Disprove : A set whose outer measure is arbitrarily small is a null set. Why does one say that Lebesgue measure is complete.

   (b) Define the notion of a Lebesgue measurable set. Show that an open set is a measurable set.

2. (a) Prove or Disprove If $\{f_n\}$ is a sequence of measurable functions then $f = \lim_{n\to\infty} f_n$ is measurable function.

   (b) Show that the product of measurable functions is a measurable function.

3. (a) State and prove Fatou's Lemma . Illustrate that strict inequality may hold.

   (b) Show that if a sequence of measurable functions converge (pointwise) to a function f dominated by an integral function g then f is integrable. Does the sequence formed by integrals of f_n converge to the integral of f in this case ? Justify.

4. (a) Show that if a sequence converges in mean then there exists a subsequence which converges almost everywhere and converges in mean

   (b) State and Prove Riesz -Fischer theorem for L^1.

5. (a) State Fubini and Tonneli's thorem for $\mathbb{R}^2$. Prove Tonneli's theorem using Fubini theorem.

   (b) Show that $\int_0^1 \frac{\log(1-x)}{x}\,dx = \sum_{n=1}^{\infty} \frac{1}{n^2}$

6. (a) State and Prove Riemann-Lebesgue lemma for lebsgue integrable functions for an unbounded interval. Does the lemma hold for improper Riemann integral? Justify your answer.

   (b) Derive Parsevel's identity. If the integral of a non-negative function is zero what can one conclude about the function? Justify.

7. (a) Prove or Disprove : If $|f|$ is integrable then f is integrable. Give an example of a function which is Lebesgue integrable but not Riemann integrable.

   (b) Show that a Riemann integrable function is Lebesgue integrable. Does the Lebesgue integral $\int_0^{\infty} \frac{\sin x}{x}\,dx$ exist? Is the function $\frac{\sin x}{x}$ a measurable function? Justify.

8. Define Fourier transform. State and prove Plancharel theorem.

MT-Con. 3596-16.

M.Sc. (Mathematics) Part – II
Differential Geometry
Paper - III
April: - 2016

QP Code : 25085

Duration:[3 Hours] [Marks: 100]

N.B. 1) All questions carry equal marks.
2) Attempt any **five** questions.

1. (a) (i) State and prove Cauchy-Schwarz inequality for an inner product space V. (5)
(ii) For any $x, y \in V$, where V is an inner product space, show that $||x|| = ||y||$ if and only if $x - y$ is orthogonal to $x + y$. (5)

(a) (i) Define Isometry and show that every isometry is the composition of an orthogonal linear operator and a translation. (5)
(ii) Define Hperplane and find the normal equation of hyperplane passing through the points $(1, 2, 1)$, $(-2, -1, 3)$ and $(2, -3, -1)$. (5)

2. (a) State and prove Picard's theorem for the existence and uniqueness of solution of $\frac{dy}{dx} = f(x, y)$ with $y(x_0) = y_0$. (10)
(b) Find approximate solution upto t^4 of the initial value problem $\frac{dx}{dt} = x + y + t$, $\frac{dx}{dt} = 4x - 2y$ with $x(0) = 1$ and $y(0) = 0$. (10)

3. (a) If $f : U \subset I\!R^3 \longrightarrow I\!R$ is a differentiable function and $a \in f(U)$ is a regualr value of f then show that $f^{-1}(a)$ is a regualr surface in $I\!R^3$ and hence show that the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ is a regular surface. (10)
(b) (i) Let S be a regular surface. Show that there exist a vector subspace of dimension two which coincides with the set of tangent vectors $T_p(S)$ for $p \in S$. (5)
(ii) Let $f(x, y, z) = (x + y + z - 1)^2$. Locate the critical points and critical values of f. (5)

4. (a) Let $\gamma(t)$ be a regular curve in $I\!R^3$. Then show that its curvature is given by $\kappa = \frac{||\ddot{\gamma} \times \dot{\gamma}||}{||\dot{\gamma}||^3}$, where $\cdot$ represent differentiation w.r.t. t and hence compute the curvature of the circular helix $\gamma(t) = (acost, asint, bt)$. (10)
(b) (i) Let γ be a regular curve in R^3 with nowhere vanishing curvature then show that the image of γ is contained in a plane if and only if torsion is zero at every point of the curve. (5)
(ii) Find arc length of the logarithmic spiral $\gamma(t) = (e^t cost, e^t sint)$ and hence or otherwise find its unit speed reparametrization. (5)

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MT-Con. 4972-16.

5. (a) State and prove the generalized Stoke's theorem for the integration of exterior forms. (10)

   (b) (i) Let $f : I\!R^3 \to I\!R$ be a function. Show that point P is a critical point of f if and only if $\frac{\partial f}{\partial x}(P) = \frac{\partial f}{\partial y}(P) = \frac{\partial f}{\partial z}(P) = 0$. (5)

   (ii) If (r, θ, z) are the cylindrical co-ordinates in $I\!R^3$ given by $x = r\cos\theta$, $y = r\sin\theta$, $z = z$. Compute the volume element $dxdydz$ in terms of cylindrical co-ordinates. (5)

6. (a) (i) State and prove Meusnier theorem. (5)

   (ii) Show that a diffeomorphism $f : S_1 \to S_2$ is an isometry if and only if for any surface patch σ_1 of S_1, the patches σ_1 and $fo\sigma_1$ of S_1 and S_2 respectively have the same first fundamental form. (5)

   (b) Calculate Gaussian curvature and mean curvature of the points of surface (10)

   $\sigma(u, v) = (f(u)cosv, f(u)sinv, g(u))$ where $\dot{f}^2 + \dot{g}^2 = 1$, $\cdot$ represent differentiation w.r.t. u.

7. (a) State and prove the Frenet-Serret formulae. (10)

   (b) (i) Prove that the unit sphere $S^2 = \{(x, y, z)/x^2 + y^2 + z^2 = 1\}$ is orientable. (5)

   (ii) Is the map from the circular half-cone $x^2 + y^2 = z^2$, $z > 0$ to the xy plane given by $(x, y, z) \mapsto (x, y, 0)$ an isometry? Justify your answer. (5)

8. (a) Show that the parametrized curve $\gamma(t) = (\frac{4}{5}cost, 1 - sint, \frac{-3}{5}cost)$ is a circle and find its centre and radius. (5)

   (b) Is the curve $\gamma(t) = (\frac{1+t^2}{t}, t+1, \frac{1-t}{t})$ planar? Justify your answer. (5)

   (c) Let $\gamma(t)$ is regular curve and s be its arc length starting at any point of γ then show that γ is smooth function of t. (5)

   (d) Find the equation of tangent plane to the surface patch $\sigma(r, \theta) = (rcosh\theta, rsinh\theta, r^2)$ at $(1, 0, 1)$. (5)

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M.Sc. (Mathematics) Part – II
Graph Theory
Paper - IV April: - 2016

QP Code : 25160

External (Scheme A) (3 Hours) Total marks : 100
Internal/External (Scheme B) (2Hours) Total marks : 40

N.B. 1) **Scheme A** students answer any **five** questions.
2) **Scheme B** students answer any **three** questions.
3) **All** questions carry **equal** marks.
4) Write on **top** of your answer book the **scheme** under which you are **appearing**

1. (a) Prove that there exists two non –isomorphic realizations of the degree sequence 6,6,6,6,6,2,2,2,2,2.

(b) Prove necessary part of Erdos - Gallai conditions.

2. (a) Prove that the center of a tree consists of a single vertex or two vertices joined by an edge. Illustrate the proof.

(b) Write Fleury's algorithm for finding Eulerian trail with one example.

3. (a) State and prove Tutte's theorem about existence of perfect matching.

(b) How many perfect matchings tree can have ? Justify.

4. (a) Explain edge connectivity, vertex connectivity and minimum degree with one example each.

(b) Write Ford-Fulkerson labeling algorithm.

5. (a) Let G be a graph and u, v be non-adjacent vertices in G then prove that:

$\chi(G) = \min\{\chi(G + (u, v), \chi(G, uv)\}$.

(b) If G is a connected graph other than an odd cycle then prove that $\chi(G) \leq \Delta(G)$.

6. (a) Prove that a connected graph is isomorphic to its line graph if and only if it is a cycle.

(b) Prove that Hamiltonian closure of a graph G is well defined.

7. (a). State and prove Euler's formula for a planar graph.

(b) Prove that every planar graph G with $p \geq 4$ has at least four points of degree not exceeding 5.

8. (a) Define Ramsey Number R(p, q) for p, q $\geq$ 2. Show that

$R(p, q) \leq R(p-1, q) + R(p, q-1)$ if $p, q \geq 3$.

(b) Show that the Ramsey Number $R(3, 3, 3) \leq 17$.

QP Code : 25029

External (Scheme A) (3 Hours) **[Total Marks:100**
Internal (Scheme B) (2 Hours) **[Total Marks:40**

Note:

(1) **External (Scheme A) students answer any five questions.**

(2) **Internal (Scheme B) students answer any three questions.**

(3) **All questions carry equal marks. Scientific calculator can be used.**

(4) **Write on top of your answer book the scheme under which you are appearing.**

Que. 1 (a) Define: Absolute error and Percentage error.
Evaluate the sum $S = \sqrt{5} + \sqrt{7} + \sqrt{11}$ up to 4 significant digits and find its absolute and relative errors.

(b) Convert the decimal fraction $(211.46875)_{10}$ to the binary form and then convert to the hexadecimal form.

Que. 2 (a) Derive the Chebyshev iteration formula to find a root of the algebraic or transcendental equation $f(x) = 0$.

(b) Perform two iterations of the Newton-Raphson method to solve the system of nonlinear equations $x^2 - y^2 = 3$ and $x^2 + y^2 = 13$. Use initial approximations $x_0 = y_0 = 2.5495$.

Que. 3 (a) Solve the following system of linear equations by Gauss-Jordan method.

$$\begin{aligned} x + 2y + z &= 3 \\ 2x + 3y + 3z &= 10 \\ 3x - y + 2z &= 13. \end{aligned}$$

(b) Find the largest eigen value in magnitude and corresponding eigen vector of the following matrix by power method.

$$A = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}. \quad \text{Use} \quad X_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and take 4 iterations.}$$

Que. 4 (a) Derive Lagrange's interpolation formula for unequal intervals.

(b) Use Newton's divided difference formula to find the fourth degree curve passing through the points $(2, 3)$, $(4, 43)$, $(5, 138)$, $(7, 778)$ and $(8, 1515)$.

Que. 5 (a) Derive Newton-Cotes quadrature formula and use it to derive Simpson's three eight rule for numerical integration.

(b) Derive two-point Gaussian quadrature formula to evaluate the integral $\int_{-1}^{1} f(x)dx$.
Use this formula to evaluate $\int_0^2 \frac{1}{x^3 + x + 1} dx$.

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Que. 6 (a) Using the least-squares method, obtain the normal equations to find the values of a, b and c when the curve $y = c + bx + ax^2$ is to be fitted for the data points $(x_i, y_i), i = 1, 2, 3......., n$.

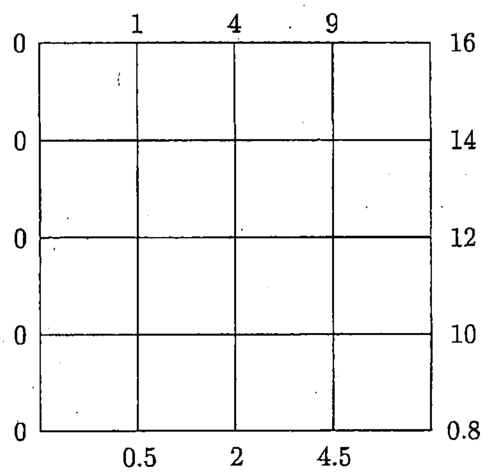
(b) Explain the term Discrete Fourier Transform (D.F.T.) and compute the (4-point) D.F.T. of the sequence $x = (1, 2, 3, 4)$.

Que. 7 (a) Derive the Adams-Bashforth corrector formula to solve the differential equation $\frac{dy}{dx} = f(x, y)$ with $y(x_0) = y_0$.

(b) Use Milne's method to compute $y(0.8)$ and $y(1)$, given that $\frac{dy}{dx} = 1 + y^2$ with $y(0) = 0,\ y(0.2) = 0.2027,\ y(0.4) = 0.4228,\ y(0.6) = 0.6841$.

Que. 8 (a) Derive a numerical method (Crank-Nicolson's method) to obtain the numerical solution of one dimensional heat equation with initial and boundary conditions.

(b) Use Liebmann's method to solve the Laplace equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ at the interior mesh points of the square region with boundary values given in the following figure.



[Take 2 iterations and obtain result correct upto three decimal places.]

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M.Sc. (Mathematics) Part – II

Functional Analysis

April: - 2016

QP Code : 25134

Scheme A (External)]	(3 Hours)	**[Total Marks:100**
Scheme B (Internal)]	(2 Hours)	**[Total Marks: 40**

Instructions:

- **Scheme A** students should attempt **any five** questions.
- **Scheme B** students should attempt **any three** questions.
- **All** questions carry **equal marks**.
- **Mention** clearly the **Scheme** under which you are appearing.

1. (a) State and prove the Minkowski's inequality for non-negative real numbers $a_1, a_2, \ldots, a_n, b_1, \ldots, b_n$.

   (b) Show that $\| (x_1, x_2, \ldots, x_n) \| = |x_1| + |x_2| + \ldots + |x_n|$ defines a norm on $\mathbb{R}^n$.

2. (a) Show that every Hilbert space is reflexive.

   (b) Show that a subspace Y of a Hilbert space H is closed iff $Y = (Y^{\perp})^{\perp}$.

3. (a) Let X be a normed linear space and Y be a closed subspace. Define $\| x + Y \|_Q = \inf \{\| x + y \| : y \in Y\}$ for each coset $x + Y$. Prove that $\| \|_Q$ defines a norm on the quotient space X/Y, by verifying all the norm properties.

   (b) Show that all norms in a normed linear space which is of finite dimension are equivalent. Also show that X is complete in each of its norms.

4. (a) Prove that there exists a linear homeomorphism from a real normed linear space of dimension n onto $\mathbb{R}^n$.

   (b) Show that l^p is separable if $1 < p < \infty$.

5. (a) State and prove Hahn Banach theorem.

   (b) Show that any linear functional on $\mathbb{R}^n$ is continuous.

6. (a) Show that the composite of two bounded linear operators, atleast one of which is compact, is compact.

   (b) Let $T : X \to Y$ be a bounded operator. Define T^* the adjoint of T. Prove that $\| T^* \| = \| T \|$ and prove further that T^* is one-one iff range of T is dense in Y.

7. (a) Let H be a Hilbert space. Show that $U : H \to H$ is unitary iff U is bijective and is an isometry.

   (b) State and prove Bessel's inequality in a Hilbert space H.

8. (a) Show that the dual space of l^1 is l^{∞}.

   (b) State and prove Uniform Boundedness theorem.
